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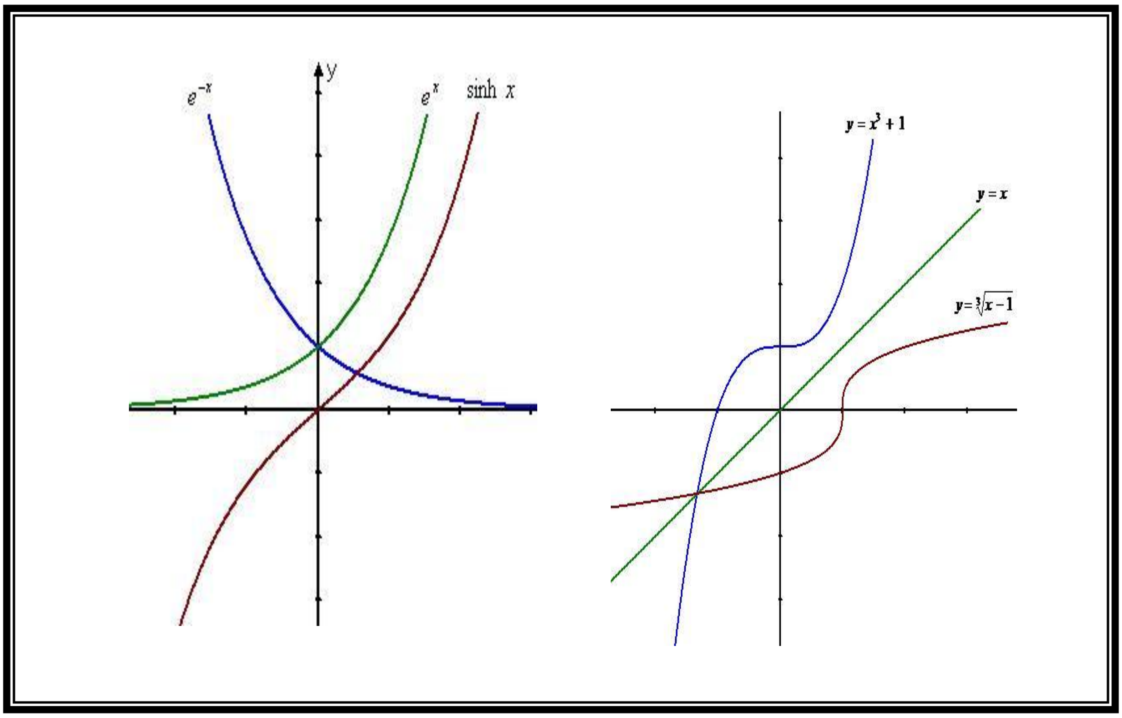

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CHAPTER 1

GENERAL FORM OF A GRAPH FOR EQUATION OF SECOND ORDER (IN PLANE)

A brief study for geometric representation of second order algebraic equation of two variables in xy-plane (Cartesian coordinates) introduced in this chapter .

First section deals with general form for equation of a circle in Cartesian coordinates (xy-plane) , and the next sections deals with equations of conic sections in general and special forms also in xy-plane .

A second order algebraic equation of two variables in xy-plane has a general form :

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Where ***at least one*** of the coefficient A , B , and C not equal zero.

The above formula can be firstly simplified by choosing $B = 0$ to get the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad \textbf{(1.0)}$$

The coefficient of equation (1.0) play an important role in the graph shape (circle , parabola , ellipse , hyperbola) as discussed later , ***depends mainly on the values and the signs of these coefficients*** .

Equation (1.1) is called the general second order equation form for graph.

The coefficient types and graphs can be summarized as follow:

If $A = C \neq 0$ with same sign , then (6.0) represent a circle
If $A \neq C$, then (6.0) represent a a conic section : with
** $A = 0$ or $C = 0$ is a parabola .
** $A \neq C \neq 0$ with same sign ellipse .
** $A \neq 0 , A \neq 0$ with different sign hyperbola .

Now a brief study for each graph will be discussed according to the coefficient constants **A , C , D** and **E** .

I- Graph Of A Circle In Cartesian Coordinates

A Circle is a plane curve consisting of the set of all points at a given fixed distance ***(<u>called the radius</u>)*** from a given fixed point ***(<u>called the center</u>)*** . If $r > 0$ is the radius and (p, q) is the center , and if (x, y) is arbitrary point on the circle (see fig. 1.1) , then by using the distance formula we can write the defining condition as :

$$\sqrt{(x-p)^2 + (y-q)^2} = r$$

Or

$$(x-a)^2 + (y-b)^2 = r^2 \qquad \textbf{(1.1)}$$

Equation (1.1) represent ***<u>equation of the circle in standard form of center</u>*** $(p\,,q)$ and radius r (see fig 1.1).

If the center of the circle be $(0\,,0)$ then equation (1.1) simplify to the form

$$x^2 + y^2 = r^2 \qquad \textbf{(1.2)}$$

Which represent ***<u>the equation of a circle in its standard form has</u>*** a center $(0\,,0)$ and a radius r (see fig.1.2) .

By squaring the terms on the left of (1.1) and re-arranging , this equation can be written in the form :

$$x^2 + y^2 + Dx + Ey + F = 0 \qquad \textbf{(1.3)}$$

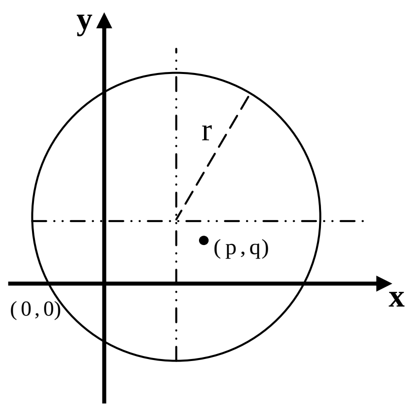

fig (1.1)

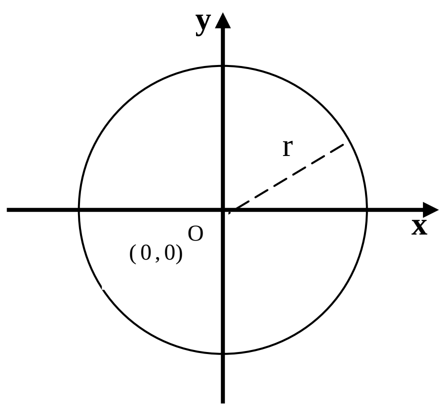

fig (1.2)

<u>N O T E</u> :

By completing the square on the x and y terms , any equation of form (1.3) can be written in the form (1.1) , therefore ; as a result of the fact the constant r^2 of (1.1) classify the following **:**

** If : $r^2 > 0$ (1.1) represent equation of a circle **.**

** If : $r^2 = 0$ (1.1) represent equation of a single point .

** If : $r^2 < 0$ (1.1) represent the empty set

<u>Example</u> :1

Find the graph type represent the equation : $3x^2 + 3y^2 - 27 = 0$.

<u>Solution:</u>

The above equation can be re-written as $x^2 + y^2 = 3^2$ which apply the standard form (1.2) i.e.,

<u>Represent a circle</u> , as coefficient of x^2 = coefficient y^2

<u>Has a center</u> $(0,0)$, as coefficient of $x = 0$, coefficient $y = 0$

<u>Has a radius</u> $r = 3$, as the given equation can represented by : $x^2 + y^2 = 3^2$

i.e. the graph represent a circle with center $(0,0)$ and radius $r = 3$. $$

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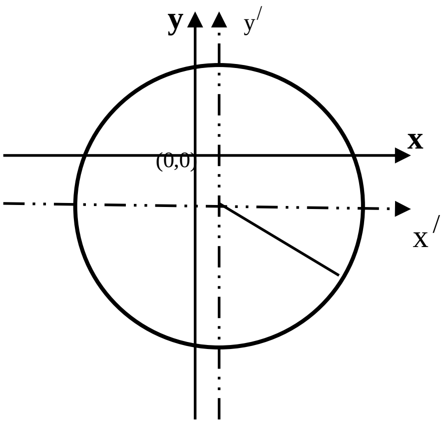
<u>Example</u> :2

Draw the graph of the equation : $x^2 + y^2 - 2x + 4y - 20 = 0$.

<u>Solution:</u>

Compare the given equation with equation (1.2.a) we get that it represent a circle with center (a,b) and radius r .



* To draw the graph , transform it to the standard form (1.2.b) as follow :

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

Complete square roots will used as :

$$\Rightarrow \quad (x^2 - 2x + 1) + (y^2 + 4y + 4) - 1 - 4 - 20 = 0$$

$$\Rightarrow \quad (x-1)^2 + (y+2)^2 = 5^2$$

which represent a circle with center (1,-2) and radius $r = 5$.

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$$$$$$$$ ***Tray by your self*** $$$$$$$$

Discus each of the following equation and draw the graph (if possible) :

1) $36(x^2 + y^2) - 24x + 180y + 193 = 0$, 2) $36(x^2 + y^2) - 24x + 180y + 229 = 0$

3) $36(x^2 + y^2) - 24x + 180y + 235 = 0$, 4) $x^2 + y^2 + 6x - 4y + 14 = 0$

5) $x = -5 + \sqrt{40 - 6y - y^2}$.

$$$$$$$$

I-1 Relative Position Of A Circle And Straight Line In Plane :

This section discuss the relative position of straight line with respect to a circle lies in its same Cartesian plane .

This positions under discussion summarized in :

** *Straight line **intersect** with the circle **in two points*** .

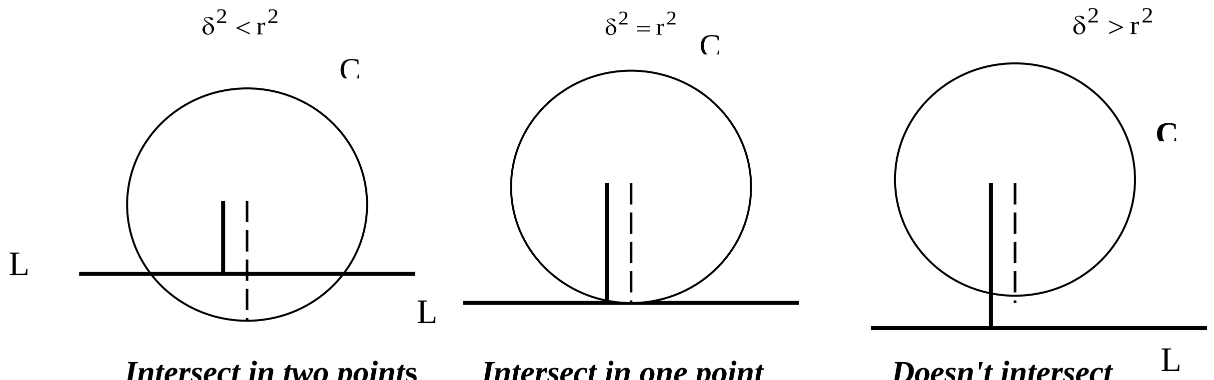
** *Straight line **intersect** with the circle **in one point**(**Tangent line**)* .

** *Straight line **doesn't intersect** with the circle **any where*** .

Let **C** be a circle satisfy equation (1.0) with $A = C = 1$ for simplicity of calculation only , i.e. : $x^2 + y^2 + Dx + Ey + F = 0$

and let **L** be a straight line has the equation $L : ax + by + c = 0$.

The relative position of the straight line can be discussed by determine the perpendicular distance between circle center and a point lies on the straight line (i.e. short distance between two points) . Let δ be that distance , then there is three relative relations between r (circle radius) and δ (the perpendicular distance) summarized as: ($\delta < r$, $\delta = r$, $\delta > r$) and we get the three following possibility as in fig (3) :



Intersect in two points ***Intersect in one point*** ***Doesn't intersect***

fig (1.3)

I-2 Equation Of A Tangent And Equation Of Perpendicular Line:

The tangent equation to the circle at a circumference point $M_\circ(x_\circ, y_\circ)$ is :

$$(x - x_\circ)(x_\circ - p) + (y - y_\circ)(y_\circ - q) = 0 \qquad \textbf{(1.4)}$$

The equation of perpendicular line to the circle at a circumference point $M_\circ(x_\circ, y_\circ)$ is :

$$(y - y_\circ)(x_\circ - p) + (x - x_\circ)(y_\circ - q) = 0 \qquad \textbf{(1.5)}$$

Example :3

Find the equation of the tangent and the equation of the perpendicular line for the circle : $x^2 + y^2 + 5x - 6y - 21 = 0$ at the point $M_\circ(2,-1)$.

Solution:

To find the required equations we must first calculate the center of the circle as $(p,q) = \left(-\frac{D}{2}, -\frac{E}{2}\right)$ and the tangent point $M_\circ(x_\circ, y_\circ)$

By using the standard form (1..3) we find hat : the center point $(p,q) = ((-5/3), 3)$, and the tangent point $M_\circ(x_\circ, y_\circ) = M_\circ(2,-1)$.

Then the equation of the tangent (by *using 1.4*) is :

$$(x-2)[2-(-5/2)] + (y+1)[-1-(3)] = 0$$

$$\Rightarrow \quad \frac{9}{2}(x-2) - 4(y+1) = 0 \quad \Rightarrow \quad 9(x-2) - 8(y+1) = 0$$

Then the tangent equation is : $9x - 8y - 26 = 0$

In similar way

The equation of the perpendicular line (by *using 1.5*) is :

$$(y+1)[2-(-5/2)] - (x-2).[-1-(3)] = 0$$

$$\Rightarrow \quad \frac{9}{2}(y+1) + 4(x-2) = 0 \quad \Rightarrow \quad 9(y+1) + 8(x-2) = 0$$

Then the perpendicular line equation is : $9y + 8x - 7 = 0$

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$$$$$$$$ ***Tray by your self*** $$$$$$$$

Find the equation of the tangent and the equation of the perpendicular line for the circle : $x^2 + y^2 - 25 = 0$ at the point $M_\circ(3,-4)$.

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Example :4

Find the equation of the circle C has the two points $A(-3,2)$ and $B(1,4)$ as end points of one of its diagonals .

Solution:

To determine circle equation it must determine the center and the radius values. (See fig.1.4) we find that :

center coordinate is the midpoint of the diagonal ends calculate as

$$(p,q)=\left(\frac{-3+1}{2},-\frac{2+4}{2}\right)=(-1,3)$$

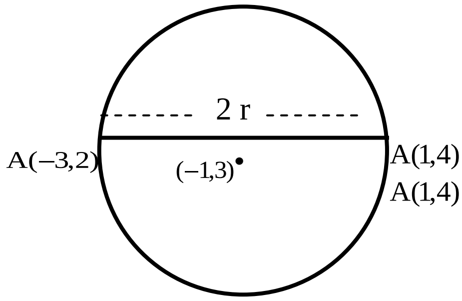

fig. (1.4)

and the redius can be calculated as:

$$\overline{AB}=2r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{(1+3)^2+(4-2)^2}=\sqrt{20}$$

and then $2r=2\sqrt{5} \quad \rightarrow \quad r=\sqrt{5}$

Then the equation of the required circle is : $(x+1)^2+(y-3)^2=5$

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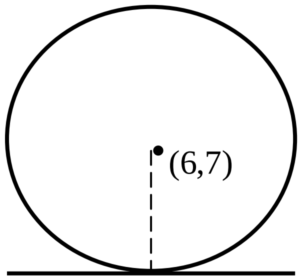
Example :5

Find the equation of the circle C has center $(6,7)$ and has a tangent line $5x-12y-24=0$.

Solution:

Note

That the distance between the point $(x_\circ,y_\circ)$ and the straight line $L: ax+by+c=0$ calculated by $\delta=\frac{|ax_\circ+by_\circ+c|}{\sqrt{a^2+b^2}}=r$, then the radius of the circle is : $r=\delta=\frac{|5(6)-12(7)-24|}{\sqrt{(5)^2+(-12)^2}}=\frac{78}{13}=6$



and the equation of the circle is : $(x-6)^2+(y-7)^2=36$.

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Example :6

Find the equation of the circle C has y-axis as its tangent line at the point (0,4) and intersect 6 units from x-axis (see fig 1.5)

Solution:

The circle equation determined as the center and the radius determined , **but in that case** an information lag has occurred so , we use the general form of circle equation (1.3) : $x^2+y^2+Dx+Ey+F=0$ and try to find the coefficient constants D, E, F as the point (0,4) lie on circle circumference then it satisfy (1.3) (note that $x=0$) we get :

In y- axis($x=0$) then (1.3) leads to

$$y^2+Ey+F=0 \qquad (1)$$

which represent equation of 2^{nd} order has two equal roots as it tang y-axis at (0,4) and has the form :

$$(y-4)^2=0 \rightarrow \quad y^2-8y+16=0 \qquad (2)$$

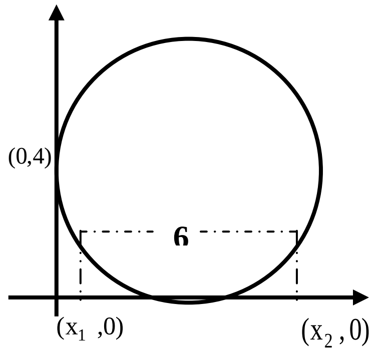

fig. (1.5)

Compare with equation (1) with (2) we get :

$$E=-8 \text{ and } F=16$$

Then substitute by this value (1.3) we get :

$$x^2+y^2+Dx-8y+16=0 \qquad (3)$$

To find the value of coefficient D , put $y=0$ in equation (3) to find the point of intersection of a circle with x-axis , we get :

$$x^2+Dx+16=0 \qquad (4)$$

Which represent also equation of 2^{nd} order has two roots

$$x_1=\frac{-D+\sqrt{D^2-64}}{2} \quad , \quad x_2=\frac{-D-\sqrt{D^2-64}}{2}$$

But as the intersect length of x-axis (x_2-x_1) is equal to **6** {i.e. $(x_2-x_1)=6$} ,

then : $(x_2-x_1)=-\sqrt{D^2-64}=6 \Rightarrow \ D^2-64=36 \ \Rightarrow \ D^2=100 \ \Rightarrow \ D=\pm 10$,

and then , the circle standard form is : $x^2+y^2\pm 10x-8y+16=0$, which represent the standard form of the two circle :

$$(x\pm 5)^2+(y-4)^2=25 \ .$$

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I -3 Intersect Of Tow Circles In Plane :

If two intersect circles in plane with two different centers , then its equations has the forms :

$$\left.\begin{array}{l} C_1 : x^2 + y^2 + D_1x + E_1y + F_1 = 0 \\ C_2 : x^2 + y^2 + D_2x + E_2y + F_2 = 0 \end{array}\right\} \qquad (1.6)$$

And to find its intersection point the corresponding system (1.7) must be solved :

$$\left.\begin{array}{l} x^2 + y^2 + D_1x + E_1y + F_1 = 0 \\ (x^2 + y^2 + D_2x + E_2y + F_2) - (x^2 + y^2 + D_1x + E_1y + F_1) = 0 \end{array}\right\} \qquad (1.7)$$

Example :7

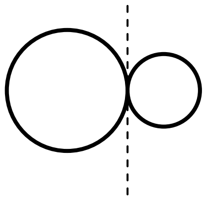
Discus the intersection of the two given circles formatted as follow :

$$\left.\begin{array}{l} C_1 : x^2 + y^2 - 2x - 4y + 4 = 0 \\ C_2 : x^2 + y^2 - 10x - 4y + 20 = 0 \end{array}\right\}$$

Solution:

The given equation of the form (1.6) and transform to the equivalent system (1.7) as :

$$\left.\begin{array}{l} x^2 + y^2 - 2x - 4y + 4 = 0 \\ (x^2 + y^2 - 10x - 4y + 20) - (x^2 + y^2 - 2x - 4y + 4) = 0 \end{array}\right\}$$



$$\Leftrightarrow \quad \left.\begin{array}{l} x^2 + y^2 - 2x - 4y + 4 = 0 \\ x - 2 = 0 \end{array}\right\} \qquad \textbf{(1)}$$

and by solving system (1) we find that the two circles intersect in two consides points $M(2,2)$ i.e. , two tangent circles at that point .

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Example :8

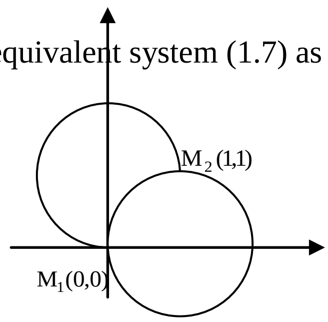
Discus the intersection of the two given circles formatted as follow :

$$\left.\begin{array}{l} C_1 : x^2 + y^2 - 2x = 0 \\ C_2 : x^2 + y^2 - 2y = 0 \end{array}\right\}$$

Solution:

The given equation of the form (1.6) and transform to the equivalent system (1.7) as :

$$\left.\begin{array}{l} x^2 + y^2 - 2x = 0 \\ (x^2 + y^2 - 2y) - (x^2 + y^2 - 2x) = 0 \end{array}\right\}$$



$$\Leftrightarrow \quad \left.\begin{array}{l} x^2 + y^2 - 2x = 0 \\ x = y \end{array}\right\} \qquad \textbf{(1)}$$

and by solving system (1) we find that the two circles intersect in two different points $M_1(0,0)$ and $M_2(1,1)$.

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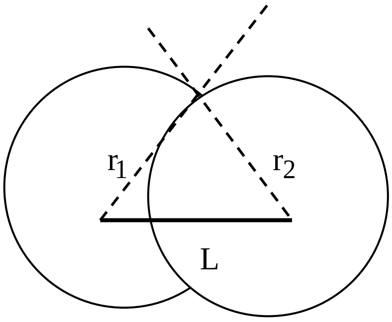
IMPORTANT RESULT



** For two intersected circles , and if its tangents at any point of intersection are perpendicular , we say that the tow circle are perpendicular and denoted by : $C_1 \perp C_2$, and according Physighrath theorem we get that : $L^2 = r_1^2 + r_2^2$, where

$L =$ the distance between its centers .
$r_1 =$ radius of the first circle .
$r_2 =$ radius of the second circle .

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E x e r c i s e (1–1)
C i r c l e

I- Select (as soon as you look) the graph types for each of the following equations :

1) $2x^2 - 3x + 2y^2 + 2y = 0$, 2) $x^2 - 2y^2 + x - 5y + 12 = 0$.

3) $y^2 + 2x + y = 12$, 4) $3x^2 + 12y^2 + 12x + 21 = 0$.

5) $3x + 2y - 5 = 0$, 6) $5x^2 - 5y^2 + 2x - 10y = 25$.

II - Find the standard form for the circle C satisfy the following knowledge :

1) Passing through the origin $(0,0)$, x-axis is its diagonal with radius $r = 5$.

2) Has a radius $r = 4$, tangent to the two axis's and lies in 1st quadrant .

3) Passing through the three points $M_1(0,2)$; $M_2(1,1)$; $M_3(2,-2)$.

4) Tang x-axis at a point $(5,0)$, and intersect 10 units of y-axis .

II- Find the center and the radius of each of the following circle and draw the graph for each one :

1) $x^2 - 6x + y^2 + 4y = 23$, 2) $x^2 + y^2 + 10x - 4y + 13 = 0$.

3) $(x+3)^2 + x^2 = 9$

III- Discuss the relative position for the graph : $x^2 - 12x + y^2 - 14y + 49 = 0$

with respect to each of the following straight lines :

1) $5x - 12y - 37 = 0$, 2) $5x - 12y - 24 = 0$,

3) $(x+3)^2 - x^2 = 9$.

IV- Find the equation for both the tangent line and the perpendicular line for each of the following circles at the given point :

1) $x^2 + y^2 + 4y - 12 = 0$; $(2,-3)$.

2) $x^2 + y^2 - 3x - 8y + 18 = 0$; $(0,4)$.

3) $x^2 + y^2 + 8x + 2y + 16 = 0$; $(-4,-2)$.

4) $x^2 + y^2 - 2x + y - 1 = 0$; $(1,1)$.

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II- Conic Sections

The classical Greeks – Archimedes , Apollonius and others – notes that when a plane (***not pass the cone vertex***) cut a cone the then arise graph is called ***the conic sections*** or for simply ***the conic*** .

N O T E :

** If cutting plane ***perpendicular to*** the cone then arise graph is a circle .

** If cutting plane is ***not perpendicular to*** the cone then arise graph is a ***conic*** **.**

** Three types conics arise according to the intersect position between the cone and the which called ***Parabola*** , ***Ellipse*** and ***Hyperbola*** .

In section 1-1 a detail discussion about the circle introduced , the next sections introduce detailed information about conics .

The Conic Sections

A general form of conic section fig. (1.6) is a plane curve arise from a moving point **P** such that the ratio between its distance about fixed point **F** (***called the focus of the conic)*** to the distance about fixed straight line **d *(called the directrix of the conic)*** equal to fixed value **e (*called the eccentricity*)**

i.e. ,

x is called the ***Conic Axis*** .

F is called the ***Conic Focus*** .

V is called the ***Conic Vertex***.

The straight line **d** is called the ***Conic Directrix*** .

The straight line $DD^{/}$ is called ***the Perpendicular Focus cord*** of the conic **.**

The ratio $e = (FP / NP)$ is called ***The eccentricity*** of the conic **.**

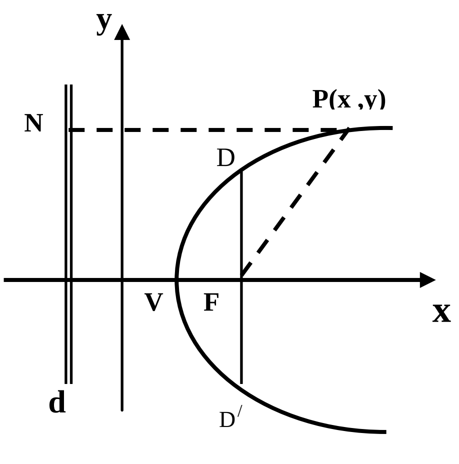


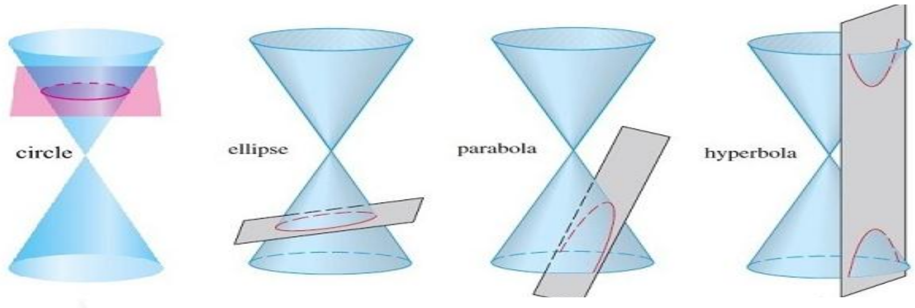
fig (6.6)

N O T E :

** Conic focus and conic vertex lies on conic axis .

** The perpendicular focus cord parallel to the conic directrix and both of them are perpendicular to the conic axis but in different sides of a vertex and of same distance of it .

The next shapes form the different intersection position between the plane and the cone , and also the different resulting graphs .



$e<1$ $e=1$ $e>1$

II-1 <u>The Parabola</u> :

($A=0$ or $C=0$ in equation (1.0))

A parabola is a plane curve consisting of the set of all points **P** that are **<u>equally distance</u>** from a given fixed point **F** (the focus) and a given fixed line **d** (called the directrix) , i.e. $e=1$, as shown in fig.(1.7.0)

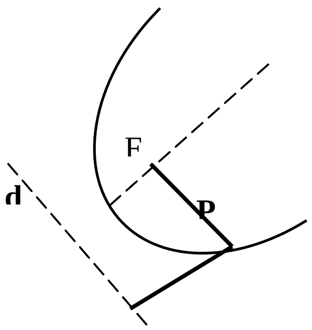


fig. (1.7.0)

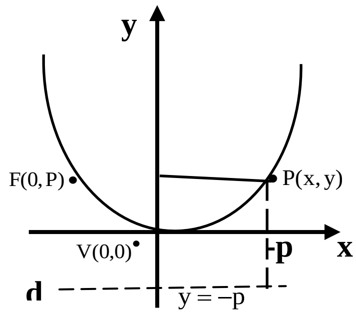


fig. (1.7.a)

<u>To find a simple equation for this curve</u> , we introduce the coordinate system as shown in fig.(1.7.a), in which the focus is the point $F(0,P)$, where **p** is a positive number (represent the distance between the focus and the vertex) and the directrix is the line $y=-p$. If $P(x,y)$ any arbitrary point on the parabola , then by using distance formula the definition condition (as $e=1\rightarrow$ the distance between the focus and the point equal to the distance between the point and the directrix) i.e. ,

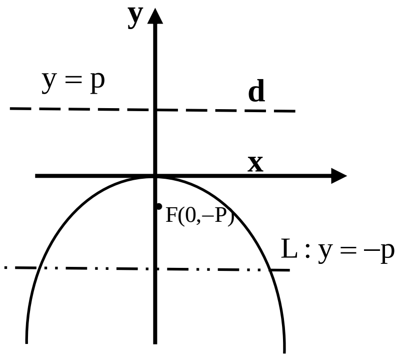
$$\underbrace{\sqrt{(x-0)^2+(y-p)^2}}_{\text{distance between focus and point}} = \underbrace{(y+p)}_{\text{distance between point and directrix}}$$

and by squaring and simplifying we get :

$$x^2+y^2-2py+p^2=y^2+2py+p^2 \quad \rightarrow \quad \boxed{x^2=4py} \qquad \textbf{(1.7.a)}$$

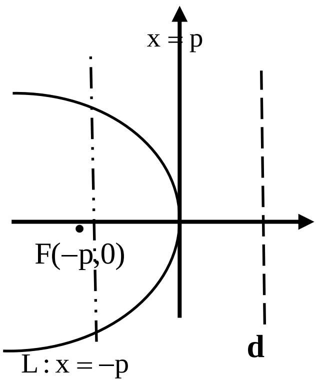
Equation (1.7.a) is therefore the equation of this particular parabola in standard form .

If we change the position of the parabola relative to the coordinate axes , we naturally change its equation. Three other simple positions , each with corresponding equation are shown in fig. (1.9)



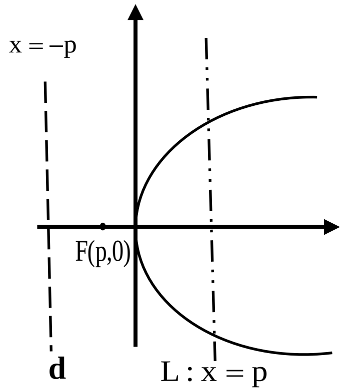
$$x^2 = -4py$$

fig. (1.7.b)



$$y^2 = -4px$$

fig (1.7.c)



$$y^2 = 4px$$

fig (1.7.d)

Next section give a general standard form of parabola in the xy-plane with a vertex $V(a,b)$ instead of $V(0,0)$ without proofs , but to prove this forms it is easy by applying the same previous procedures used in calculated equation (1.7)

<u>N O T E :</u>

The value **4p** represent ***<u>the perpendicular cord length</u>*** **<u>General Standard Form For Parabolic Equations :</u>**

(Parabola with vertex $V(a,b)$ **instead of** $V(0,0)$ **)**

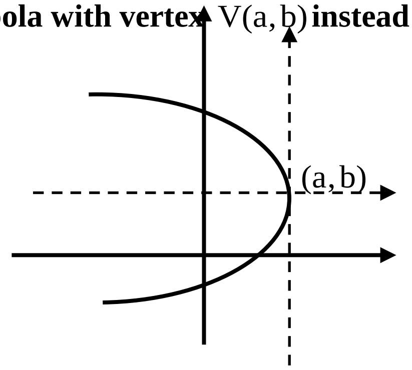


fig. (1.8.a)

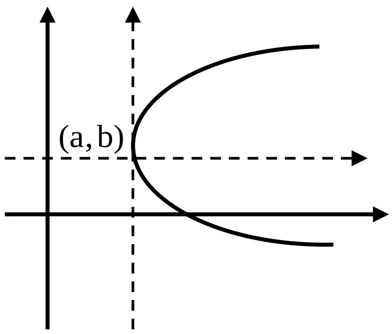
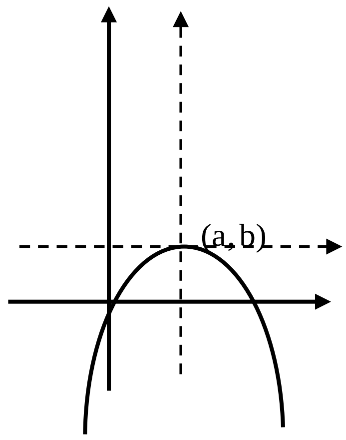


fig (1.8.b)

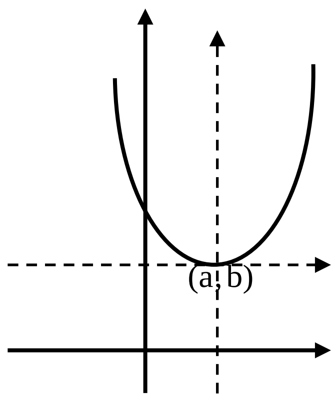
$$(y-b)^2 = -4p(x-a)$$

$$(y-b)^2 = 4p(x-a)$$



$$(x-a)^2 = -4p(y-b)$$

fig. (1.8.c)



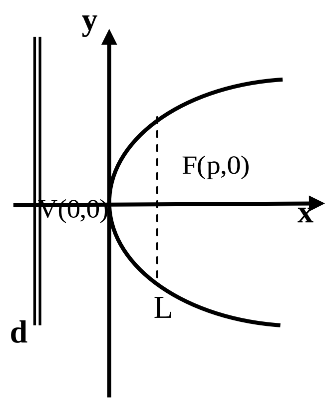
$$(x-a)^2 = 4p(y-b)$$

fig (1.8.d)

<u>Example</u> :9

Discus and draw the graph of the equation $y^2 = 8x$, and deduce all of its available .information

<u>Solution:</u>



Compare the given equation with this given with fig.(6.7.d) we get that :

* The conic axes is x-axis (the variable of 1^{st} order) and open right (refer to the $+ve$ sign of the equation) .

* Focus $F(p,0) = F(2,0)$ {as $4p = 8 \to p = 2$ }

* Vertex $V(a,b) = V(0,0)$ { as $(a,b) = (0,0)$ }

$$d : x = -2$$

* Perpendicular cord equation $L : x = 2$ and Perpendicular cord length $4p = 8$

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N O T E JUST FOR R E M E M B E R :

** The conic axes (axis of symmetry) is the axis has the variable of 1st order .

** The conic direction (open) refer to the equation sign ($+\text{ve}$ sign for right or up but $-\text{ve}$ sign for left or down) .

** Focus lies inside the cone and on the axis of symmetry and of a distance **p** from the vertex .

** Vertex lies on the axis of symmetry (conic axis).

** Directrix **d** perpendicular to the axis of symmetry and of a distance **p** from in opposite direction of the focus .

** Perpendicular cord **L** perpendicular to the axis of symmetry , of a distance **p** from the vertex , passing thought the focus **F** and of length 4p.

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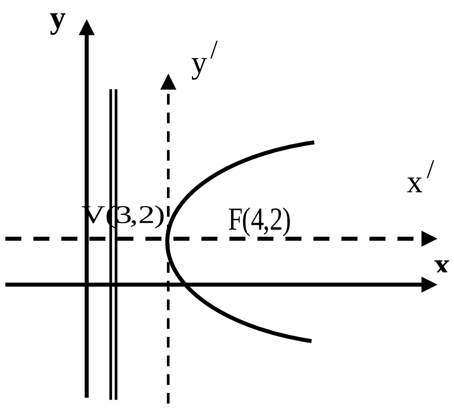
Example :10

Discus and draw the graph of the equation $y^2 - 4y - 4x + 16 = 0$, and deduce all of its available .information .

Solution



Modify the given equation by using complete square as follows:

$$y^2 - 4y - 4x + 16 = 0$$
$$\Rightarrow \quad (y^2 - 4y + 4) - 4x + 16 - 4 = 0$$
$$\Rightarrow \quad (y-2)^2 = 4x - 12$$
$$\Rightarrow \quad (y-2)^2 = 4(x-3) \qquad **$$

Compared ** with equation (1.8.b) to get the following information:

* The conic axes is $x^/ - \text{axis}$ (parallel to x-axis , corresponding to the variable of 1st order) and open right (refer to the $+\text{ve}$ sign of the equation)

* Vertex $V(a,b) = V(3,2)$

* Focus $F(p,q) = F(a+p,b) = F(4,2)$ (as $4p = 4 \rightarrow p = 1$)

* Directrix equation $d : x = 2$ $\{\text{as} : x = -a + p \Rightarrow x = -1 + 3 \Rightarrow x = 2\}$.

* Perpendicular cord equation $L : x = 4$ $\{\text{as} : x = a + p \Rightarrow x = 1 + 3 \Rightarrow x = 4\}$ and Perpendicular cord length $4p = 4$.

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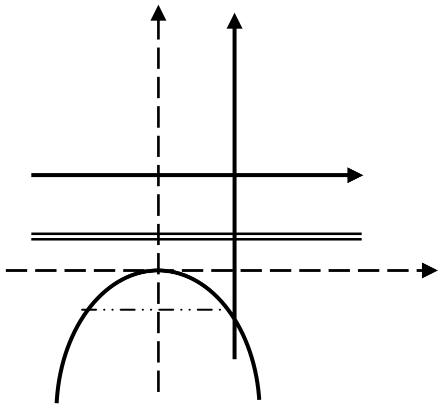
Example :11

Discus and draw the graph of the equation $x^2 + 4x + 4y + 16 = 0$, and deduce all of its available .information .

Solution:



Modify the given equation by using complete square as follows:

$\Rightarrow \quad x^2 + 4x + 4y + 16 = 0$

$\Rightarrow \quad (x^2 + 4x + 4) + 4y + 16 - 4 = 0$

$\Rightarrow \quad (x+2)^2 = -4y - 12$

$(x+2)^2 = -4(y+3) \quad$ **

Compared ** with equation (1.8.c) to get the following information :

* The conic axes is $y^{/} - \text{axis}$ (parallel to y-axis , corresponding to the variable of 1st order)and open down (refer to the $-\text{ve}$ sign of the equation)
* Vertex $V(a,b) = V(-2,-3)$
* Focus $F(p,q) = F(a, q+b) = F(-2,-3-1) = F(-2,-4)$ { as $4p = 4 \rightarrow p = 1$}
* Directrix equation $d: y = -2$ {as : $y = -a + q \Rightarrow y = -(-1) - 3 \Rightarrow y = -2$} .
* Perpendicular cord equation $L: y = -4$ {as : $y = a + q \Rightarrow y = -1 + (-3) \Rightarrow y = -4$ }
  and Perpendicular cord length $4p = 4$.

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Example :12

Deduce the standard equation of the parabola that has a vertex $V(-4,2)$ and has a directrix is the equation $y = 5$ and then draw the graph represent this parabola .

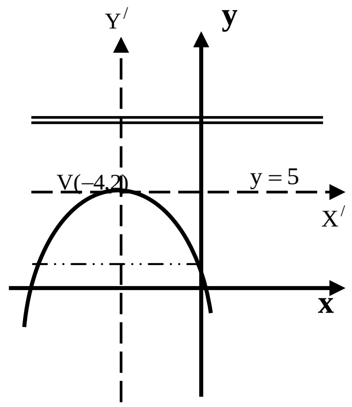
Solution :

In such problem ***it is more convenient (prefer) to represent*** the given information as a draft on the coordinate axes ***at first*** , and ***then compare with*** the suitable form of equation (1.8)

** Directrix equation $y = 5$, i.e., represent a straight line parallel to x-axis and of a distance equal 5 from it .

** Directrix is perpendicular to conic axis, then , y-axis is the conic axis .

** As the vertex $V(-4,2)$ and the directrix lies in opposite side of the focus , Then from figure geometry the parabola open down .



Then by comparing the given information and the deducing results , its clear that the figure coincide with equation (1.8.c) figure coincide with equation (1.8.c) .

So the standard equation form is :

$$(x-a)^2 = -4p(y-b) \quad ,$$

i.e. $$(x+4)^2 = -12(y-2)$$

with $(a,b) = (-4,2)$ { vertex coordinates }. and $p = 3$ { the distance between the vertex and the directrix }and the perpendicular cord length is $4p = 12$.

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***Example* :13**

Deduce the standard equation of the parabola that has a Focus $F(2,0)$ and its directrix has the equation $x = -2$ and then draw the graph represent this parabola **.**

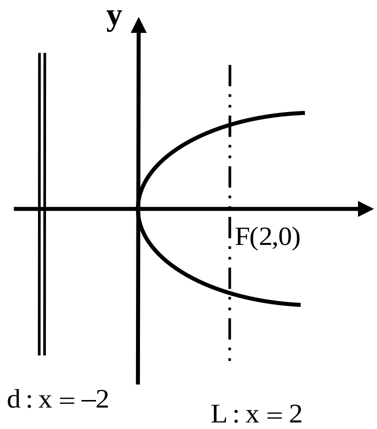
Solution :

As in example 12 ***it is more convenient (prefer) to represent*** the given information as a draft on the coordinate axes ***at first*** , and ***then compare with*** the suitable form of equation (1.8)

** Directrix equation $x = -2$, i.e., represent a straight line parallel to y-axis and of a distance equal 2 from it .

** Directrix is perpendicular to conic axis, then , x-axis is the conic axis .

** As the vertex $F(2,0)$ and the directrix lies in opposite side of the focus , then from figure geometry the parabola open right .



Then by comparing the given information and the deducing results , its clear that the figure coincide with equation (1.7.d) .

So the standard equation form is :

$$y^2 = 4px$$

i.e. $$y^2 = 8x$$

with vertex coordinates $O(0,0)$

and $p = 2$ (distance between the focus and the vertex) , so and the perpendicular cord length is $4p = 8$.

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<u>**Exercise**</u> **(1–2)**

<u>**Parabola**</u>

I- Discus and draw the graph of each of the following equation :

1)- $8x^2 = y$, 2)- $y = x^2 - 4x + 2$, 3)- $y^2 - 12 = 12x$

4)- $2x^2 = 8y$, 5)- $(x-3)^2 = 16(y-2)$, 6)- $(y+2)^2 = -20(x+2)$

7)- $12y^2 = -48x$, 8)- $(x+4)^2 = -12(y-2)$, 9)- $(y-2)^2 = 4(x-3)$

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II- Deduce the standard equation form for each of the parabola :

1)- Has a vertex $V(3,-5)$ and directrix equation $x = 2$.

2)- Has a Focus $F(0,-4)$ and directrix equation $y = 4$.

3)- Has a vertex $V(1,-2)$ and a Focus $F(1,0)$.

4)- Has a vertex $V(3,3)$ and directrix equation $y = 2$.

5)- Has a Focus $F(2,4)$ and directrix equation $x = -1$.

6)- Has a vertex $V(-1,0)$ and a Focus $F(-4,0)$.

7)- Has a Focus $F(5,3)$ and directrix equation $y = -1$.

8)- Has a vertex $V(2,2)$ and a Focus $F(3,2)$.

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III- Deduce the standard equation form for the parabola with symmetric axis parallel to y-axis , and passing thought the points $(2,1)$, $(1,2)$ and $(-1,-2)$, then draw the graph represent this parabola .

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II-2 The Ellipse :

($A \neq 0 \neq C$ in equation (1.0)**)**

An ellipse is the locus of a point P that moves in such a way that the sum of its distance from ***two fixed points*** F and $F^{/}$ constant as shown in fig. (1.9.0) (i.e. $FP + F^{/}P = 2a$)

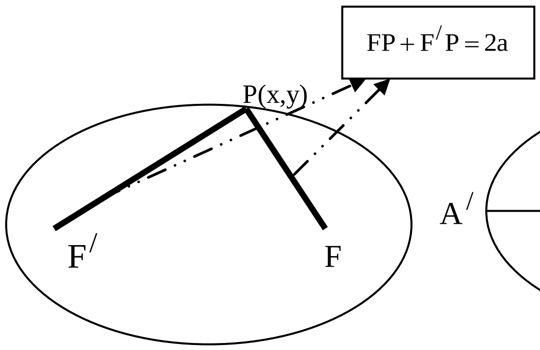


fig. (1.9.0)

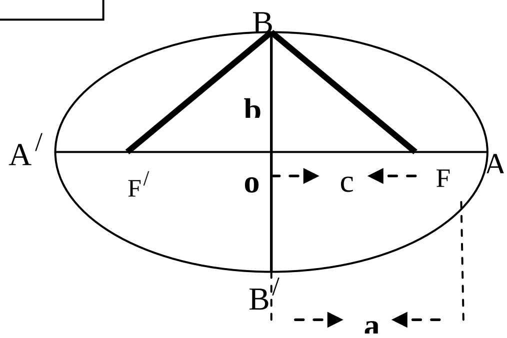


fig. (1.9.00)

A several standard notions for the dimension of the ellipse will introduced now I fig. (1.9.00).

** The two points F and $F^{/}$ are called ***the foci*** (plural of ***focus***) of the ellipse .

** The curve of symmetry $AA^{/} = 2a$ is called ***the major axis*** of the ellipse , passing through the foci and **(a)** is called the semi-major axis .

** The perpendicular bisector of the line segment $FF^{/}$ the segment $BB^{/} = 2b$ is called ***the minor axis*** of the ellipse **(b)** is called the semi-minor axis

** The two points A and $A^{/}$ at the end of the major axis are called the ***vertices of the ellipse*** .

** The distance between the foci is equal to 2c

** ***If*** the major axis coincide with x-axis , the point $o(0,0)$ is called ***the of the ellipse*** . and then the coordinates of the major points of the ellipse are corresponding to $A(a,0)$, $A^{/}(-a,0)$, $B(0,b)$, $B^{/}(0,-b)$, $F(c,0)$ and $F^{/}(-c,0)$.

From fig. (1.9.a) its clear that : $a^2 = b^2 + c^2$ (Physaghorth theorem) (i)

and it is easy to see that : $b < a$.

** The ration c/a is called ***the eccentricity of the ellipse*** and is denoted by :

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} \qquad \textbf{(1.9.0)}$$

and notice that : $0 < e < 1$.

To simplify the equation of the ellipse , and as we take x-axis as major axis fig.(1.9.a) and from fig.(1.9.0) it is clear that : $FP + F^{/}P = 2a$ and as the given coordinates point are $P(x, y)$, $F(c,0)$ and $F^{/}(c,0)$ then use the distance rule between two points then :

$$\underbrace{\sqrt{(x-c)^2 + y^2}}_{PF^{/}} + \underbrace{\sqrt{(x+c)^2 + y^2}}_{PF} = 2a \qquad \text{(ii)}$$

To simplify equation (ii) , follow the usual procedure for eliminating radicals , as :

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

By squaring both side and simplify we get :

$$PF = \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x \qquad \text{(iii)}$$

And from (iii) and the relation $F^{/}P = 2a - FP$ we get :

$$PF = \sqrt{(x+c)^2 + y^2} = a + \frac{c}{a}x \qquad \text{(iv)}$$

By squaring again and simplify ***either*** of equation (iii) or (iv) we get :

$$\left(\frac{a^2 - c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2 \quad \text{Or} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \ ,$$

Finally by putting the above equation in its final form we get :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \textbf{(1.9)}$$

Equation (1.9.a) represent the standard form for the equation of the ellipse shown as in fig.(1.9.a) specially as considered that $a > b$.

N O T E

Equation (1.9.a) : $\frac{x^2}{(\ )^2} + \frac{y^2}{(\ )^2} = 1$ with unequal denominators represents the equation of an ellipse and the equation whether the foci and major axis lies on x-axis or the y-axis which is determined by which denominator is large as shown in the following figures .

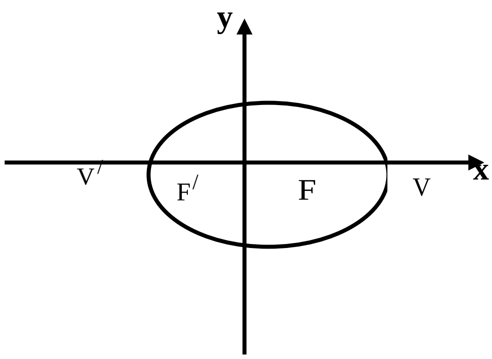


fig. (1.9.a)

$a > b$

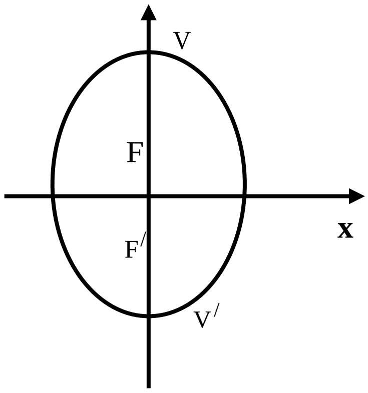


fig. (1.9.b)

$a < b$

<u>The Ellipse</u> of a Center $O^/(p,q)$:

Here we discus (without proof) the standard form of the equation of ellipse which has a center $O^/(p,q)$ (transform of coordinates) and its standard figures as follows :

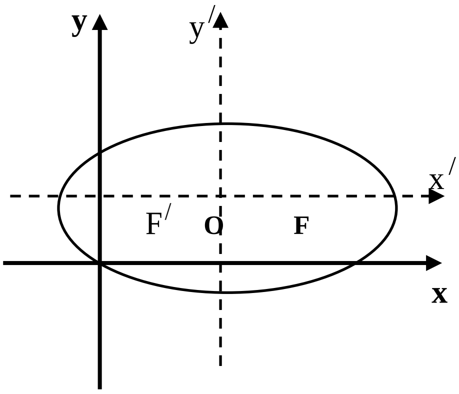


fig. (1.10.a)

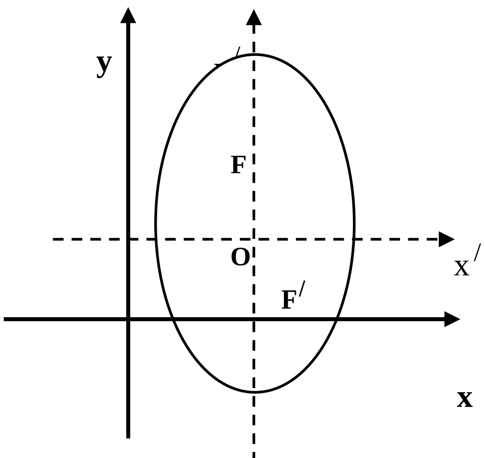


fig. (1.10.b)

$a > b$

Major axis $x^/$ parallel to x-axis .

Minor axis $y^/$ parallel to y-axis .

Foci $F(c+p,q)$ and $F^/(-c+p,q)$.

Vertices $V(a+p,q)$ and $V^/(-a+p,q)$

$b > a$

Major axis $y^/$ parallel to y-axis .

Minor axis $x^/$ parallel to x-axis .

Foci $F(p,c+q)$ and $F^/(p,-c+q)$.

Vertices $V(p,b+q)$ and $V^/(p,-b+q)$

$$\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}=1 \qquad \textbf{\textit{(1.10)}}$$

***Example* :14**

Discus and draw the graph of the equation $4x^2 + 9y^2 = 36$, and deduce all of its available .information .

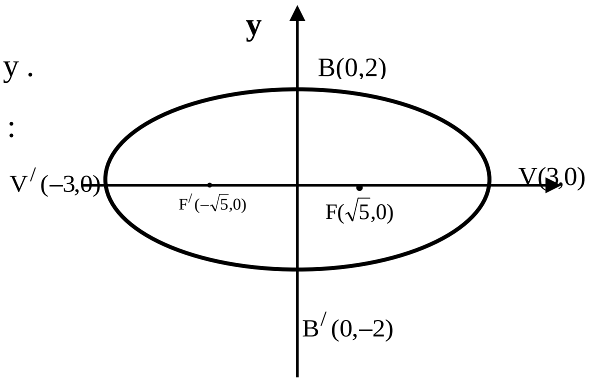
Solution:

Its clear that the equation represent equation ***of simple ellipse*** as :

** coefficient of x^2 and coefficient of y^2 are exits and different and of same sign (i.e. , $A \neq C \neq 0$) .

** has a center $O(0,0)$ as doesn't contain x or y .

Now we put the equation in its standard form as :



$$4x^2 + 9y^2 = 36 \quad \Rightarrow \quad \frac{4x^2}{36} + \frac{9y^2}{36} = 1$$

$$\Rightarrow \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad \text{(i)}$$

Compare the last equation (i) with the standard For ,f and fig.(1.9.a) we get :

** $a^2 = 9 \rightarrow a = 3$ and $b^2 = 4 \rightarrow b = 2$

** $a > b \rightarrow$ x-axis is the major axis with $2a = 6$, and
y-axis is the minor axis with $2b = 4$.

** Vertices are $V(3,0)$ and $V^{/}(-3,0)$.

** Foci are $F(\sqrt{5},0)$ and $F^{/}(-\sqrt{5},0)$.

As : $c^2 = a^2 - b^2 \rightarrow c^2 = 9 - 5 = 5 \rightarrow c = \sqrt{5}$

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***Example* :15**

Deduce the standard equation of the ellipse that has two Foci $(\pm 2,0)$ and two vertices $(\pm 4,0)$ and has origin $O(0,0)$ as its center .

Solution :

From the given information as has a center $O(0,0)$ then :

** The proposed equation form is: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (i)

** x-axis is the major axis (vertices locations) .

** the major axis length is $2a = 8$ (i.e. $\rightarrow a = 4$) .
(the distance between the two vertices are $V(4,0)$ and $V^{/}(-4,0)$)

** $c = 2$ the two foci are $F(2,0)$ and $F^{/}(-2,0)$

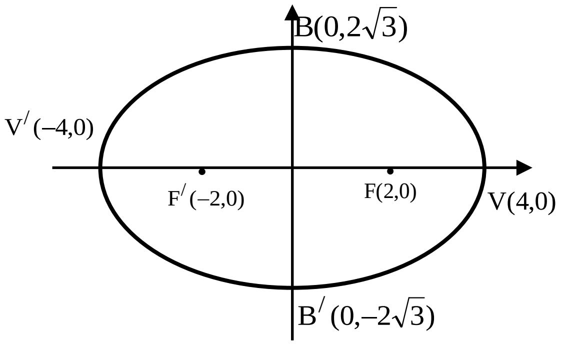
** $b^2 = 12$, as : $c^2 = a^2 - b^2 \rightarrow 4 = 16 - b^2 \rightarrow b^2 = 12$

** the minor axis length is $2b = 4\sqrt{3}$ (i.e. $\rightarrow a = 4$) .

Then refer to equation (i) with deduced information we can formulate the ellipse standard equation as:

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

:



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Example ***:16***

Discus the graph of the equation $16x^2 + 9y^2 + 64x - 18y - 71 = 0$, draw the graph and deduce all of its available .information .

Solution:

Its clear that the equation represent equation ***of general ellipse*** with vertices $O^{/}(p,q)$ as:

** coefficient of x^2 and coefficient of y^2 are exits and different and of same sign (i.e. , $A \neq C \neq 0$) .

** has a center $O^{/}(p,q)$ as coefficient of x and y both exist .

** The proposed equation form is: $\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$ (i)

To get the required equation we modify the given equation as follows :

$$16x^2 + 9y^2 + 64x - 18y - 71 = 0 \;\Rightarrow\; 16(x^2 + 4x) + 9(y^2 - 2y) - 71 = 0$$

By complete square we get :

$$\Rightarrow \quad 16(x^2 + 4x + 4) + 9(y^2 - 2y + 1) - 64 - 9 - 71 = 0$$

$$\Rightarrow \quad 16(x+2)^2 + 9(y-1)^2 = 144$$

$$\Rightarrow \quad \frac{16(x+2)^2}{144} + \frac{9(y-1)^2}{144} = 1 \;\Rightarrow\; \frac{(x+2)^2}{9} + \frac{(y-1)^2}{16} = 1 \qquad **$$

Compare equation ** with the standard form (1.10) and fig.(1.10.b) It clear that :

** ellipse center is $(p,q) = (-2,1)$

** $y^{/}$ -axis is the major axis of the ellipse (as: $b^2 = 16 \rightarrow b = 4$ and .($a^2 = 9 \rightarrow a = 3$)

** the major axis length is $2b = 8$.

** the minor axis length is $2a = 6$.

** the two foci are :

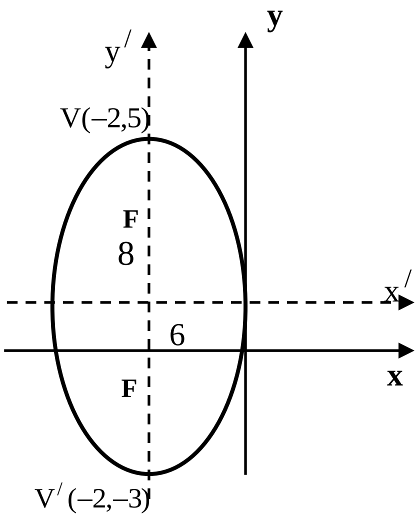
$F(p, q + c) = F(-2,1 + \sqrt{7})$ and $F^{/}(p, q - c) = F^{/}(-2,1 - \sqrt{7})$

(as : $c^2 = a^2 - b^2 \Rightarrow c^2 = 16 - 9 = 7 \Rightarrow c = \pm\sqrt{7}$)

** the two vertices are $V(p, q + b) = V(-2,1 + 4)$

and $V^{/}(p, q - b) = V^{/}(-2,1 - 4)$

i.e. $V(-2,5)$, $V^{/}(-2,-3)$



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E x e r c i s e (1–3)

E l l i p s e s

I- Discus the graphs of each of the following equations , draw the graph and deduce all of its available .information .

1)- $\frac{x^2}{25}+\frac{y^2}{16}=1$, 2)- $\frac{(x+4)^2}{49}+\frac{y^2}{25}=1$, 3)- $\frac{(x-3)^2}{9}+\frac{(y-4)^2}{36}=1$

4)- $\frac{x^2}{16}+\frac{(y+2)^2}{49}=1$, 5)- $9x^2+25y^2=225$, 6)- $x^2+2y^2+2x-20y=0$

7)- $9x^2+4y^2-54x+16y+61=0$, 8)- $4x^2+9y^2-32x-36y+64=0$

9)- $9x^2+16y^2+54x-32y-47=0$, 10)- $4x^2+9y^2+24x+18y+9=0$

II- Deduce the standard equation of the ellipse that has the following information and Deduce all available other unmentioned ellipse information

1)- Two foci $(0,\pm 2)$ and two vertices $(0,\pm 7)$ center $(0,0)$.

2)- Two foci $(\pm 5,0)$ and two vertices $(\pm 8,0)$.

3)- Two foci $(\pm 3,0)$ and minor length axis equal 2 .

4)- One of its focus $(0,2)$ and major length axis equal 10 .

5)- Center $(2,2)$ one of its focus $(-1,2)$ and major length axis equal $2\sqrt{10}$.

6)- Two Foci $(2,5)$, $(-4,5)$ and minor length axis equal 8 .

7)- Center $(3,-3)$, major axis parallel to x-axis and major length axis equal 20 and minorlength axis equal 16 .

8)- Two vertices $(0,\pm 6)$ and pass thought the point $(3,2)$.

9)- Pass through the two points $(3,2)$ and $(6,1)$.

10)- Minor axis ends are $(2,1)$, $(2,-7)$ and the distance between its foci 1.

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II-3 <u>The Hyperbola</u> :

($A \neq 0 \neq C$ in equation and in different sign in (1.0)**)**

A hyperbola is the locus of a point P that moves in such a way that the difference of its distance from two fixed points F and $F^{/}$ (called the foci)is constant .

If this constant is denoted by 2a , with $a > 0$, then a little though will show the locus consists of two branches as shown in Fig.(1.11.a) , where :

** The right branch is the locus of the equation : $PF^{/} - PF = 2a$; and

** The left branch is the locus of the equation : $PF - PF^{/} = \pm 2a$. (1)

** The defining condition for the complete hyperbola can be therefore be written as : $PF^{/} - PF = \pm 2a$.

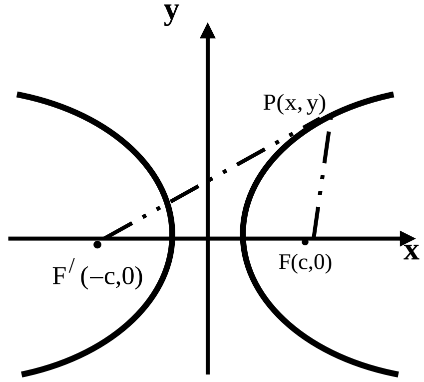


fig.(1.11.a)

<u>To find a simple equation for the hyperbola</u>, take the x-axis along the segment $FF^{/}$ and the y-axis as the perpendicular bisector of this segment .

If 2c denotes the distance between F and $F^{/}$, then $F = (c,0)$ and $F^{/} = (-c,0)$ as shown in Fig.(1.11) and (1) becomes $PF - PF^{/} = \pm 2a$

$$\Rightarrow \quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$$

By moving the second radial to the right side , squaring , and simplifying , we obtain the local radius formulas

$$\Rightarrow \quad PF = \sqrt{(x-c)^2 + y^2} = \pm\left(\frac{c}{a}x - a\right) \qquad (2)$$

and

$$\Rightarrow \quad PF^{/} = \sqrt{(x+c)^2 + y^2} = \pm\left(\frac{c}{a}x + a\right) \qquad (3)$$

where (3) follow from (2) because $PF^{/} = \pm 2a + PF$. As in (1) , *<u>the plus signs</u>* here correspond to *<u>the right branch</u>* of the curve , and the *<u>minus sign</u>* to *<u>the left branch</u>* .By squaring and simplifying , either of this equations gives

$$\left(\frac{c^2 - a^2}{a^2}\right)x^2 - y^2 = c^2 - a^2 \qquad \text{; then put } (c^2 - a^2) = b^2$$

we get :

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \qquad \textbf{(1.11.a)}$$

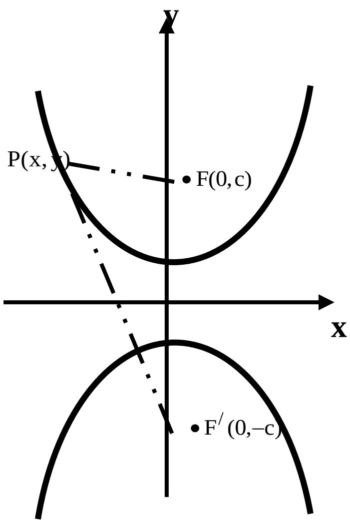
which represent the standard form of the equation of the hyperbola shown in Fig. (1.11)

Now another form of equation (1.11.1) can be evaluated ***if we replace the coefficient signs*** of both x^2 and y^2 which can be represents in its standard form as :

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \qquad \textbf{(1.11.b)}$$

and its graph seen like we rotate Fig. (1.11.a) by 90° to be as in Fig. (1.11.b) bellow :

Now we turn to a careful consideration of the hyperbola shown in Fig.(6.11.a) on the nature of the hyperbola it represents . Our discussion will reveal additional features of the hyperbola that are not obvious from the definition and that are indicated in greater detail in Fig. (6.11.c)



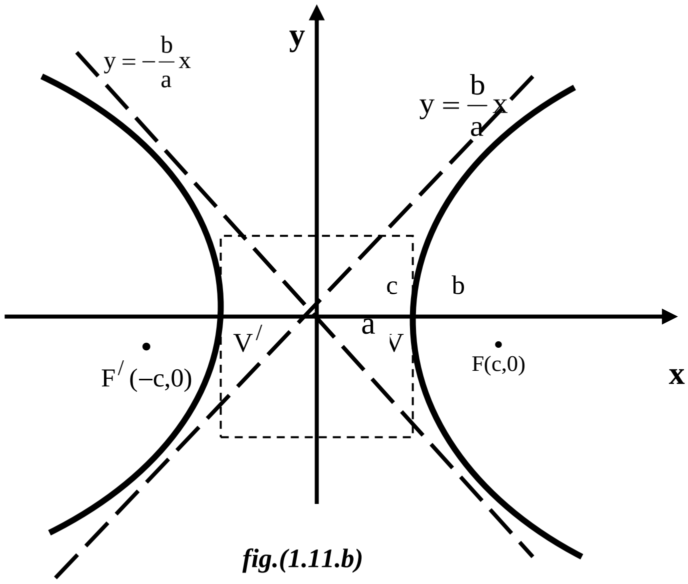


fig.(1.11.b)

<u>where :</u>

** its clear in that case the eccentricity $e > 1$, and as in ellipse case $e = (c/a)$.

** $y = \pm(b/a)x$ are a straight lines called the right and left ***asymptotes***.

** x-axis is the ***major axis*** and y-axis is ***conjugate axis*** .

** $V(a,0)$ and $V^{/}(-a,0)$ are the two vertices .

N O T E

Just as in the case of ellipse , can easily write the equation of hyperbola with center (p,q) and principal axis parallel to one of the coordinate axis .

The equations are :

$$\frac{(x-p)^2}{a^2}-\frac{(y-q)^2}{b^2}=1 \quad \textbf{Or} \quad \frac{(y-q)^2}{b^2}-\frac{(x-p)^2}{a^2}=1$$

(1.12.a) **(1.12.b)**

Example :17

Discus the graph of the equation $9x^2-4y^2=36$, draw the graph and deduce all of its available .information .

Solution:

Its clear that the equation represent equation ***of hyperbola*** with x-axis as a major axis $(+\text{ve sign})$ and y-axis is the conjugate axis $(-\text{ve sign})$.

Equation must put in the hyperbola standard form as:

$$\frac{9x^2}{36}-\frac{4y^2}{36}=1 \Rightarrow \frac{x^2}{4}-\frac{y^2}{9}=1$$

compare the given equation with the standard form (1.11.a) we get :

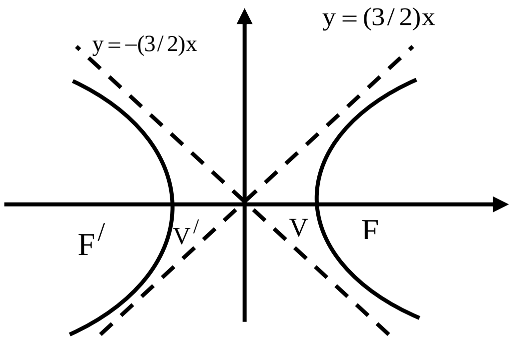
** $a=2$, $b=3$

** Vertices : $V(2,0)$, $V^{/}(-2,0)$

** Asymptotes : $y=\pm(3/2)x$,

** Foci : $F(\sqrt{13},0)$, $F^{/}(-\sqrt{13},0)$

as : $c^2=a^2+b^2=4+9=13$



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Example :18

Discus the graph of the equation $9y^2-4x^2=36$, draw the graph and deduce all of its available .information .

Solution:

Its clear that the equation represent equation ***of hyperbola*** with y-axis as a major axis $(+\text{ve sign})$ and x-axis is the conjugate axis $(-\text{ve sign})$.

Equation must put in the hyperbola standard form as:

$$\frac{9y^2}{36}-\frac{4x^2}{36}=1 \Rightarrow \frac{y^2}{4}-\frac{x^2}{9}=1$$

compare the given equation with the standard form (1.11.a) we get :

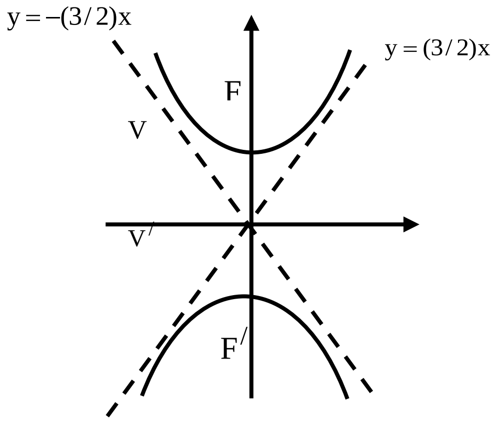
** $a=3$, $b=2$

** Vertices : $V(0,2)$, $V^{/}(0,-2)$

** Asymptotes : $y=\pm(3/2)x$,

** Foci : $F(0,\sqrt{13}), F^{/}(0,-\sqrt{13})$

as : $c^2=a^2+b^2=4+9=13$



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Example :19

Discus the graph of the equation $9x^2-4y^2-54x-16y+29=0$, draw the graph and deduce all of its available .information .

Solution:

Its clear that the equation represent equation ***of hyperbola*** with $x^{/}$ (parallel to x-axis) as a major axis ($+$ve sign) and $y^{/}$ (parallel to y-axis) is the conjugate axis ($-$ve sign) and has center (p,q) as contain (x , y of 1st order).

Equation must put in the hyperbola standard form(1.12.a) as:

$\Rightarrow \quad 9x^2-4y^2-54x-16y+29=0$

$\Rightarrow \quad (9x^2-54x)-(4y^2+16y)+29=0$

and by complete square

$\Rightarrow \quad 9(x^2-6x)-4(y^2+4y)+29=0$

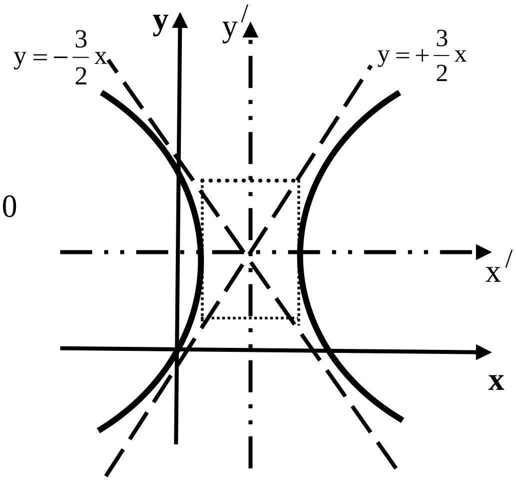
$\Rightarrow \quad 9(x^2-6x+9)-4(y^2+4y+4)+29=0$

$\Rightarrow \quad 9x^2-4y^2-54x-16y+29=0$

$\Rightarrow \quad 9(x-3)^2-4(y+2)^2-81+16+29=0$

$\Rightarrow \quad 9(x-3)^2-4(y+2)^2=36$

$\Rightarrow \quad \frac{(x-3)^2}{4}-\frac{(y+2)^2}{9}=1$ **



Compare the given equation with the standard form (1.12.a) we get :

as $a^2=4\rightarrow a=2$ and $b^2=9\rightarrow b=3$

** Hyperbola of a center $(3,-2)$.

** Vertices : $V(3+2,-2)$, $V^{/}(3-2,-2)$ i.e. $V(5,-2)$, $V^{/}(1,-2)$

** Foci $F(3+\sqrt{13},-2)$, $F^{/}(3-\sqrt{13},-2)$ (as: $c^2=a^2+b^2\Rightarrow c^2=9+5=13\Rightarrow c=\sqrt{13}$)

** Asymptotes : $y=\pm(3/2)x$.

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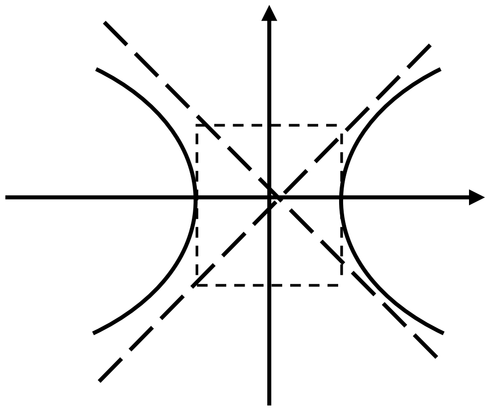
Example :20

Deduce the standard equation of the hyperbola that has a center $O(0,0)$, vertices $V(\pm 3,0)$ and pass through the point $P(5,2)$ and find all available information .

Solution :

The vertex coordinates indicate that x-axis is the major axis , y-axis is the conjugate axis and parabola has center $O(0,0)$, then has the standard form (1.11.a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

**** To find a and b**



****** $V(\pm a,0) = (\pm 3,0) \Rightarrow \quad a=3$

And as the conic pass through the point $P(5,2)$ then it verify its equation and so

$$\frac{5^2}{9}-\frac{2^2}{b^2}=1$$

$\Rightarrow \qquad b^2=\frac{9}{4} \Rightarrow b=\frac{3}{2}$, and then :

** The standard form of the conic is : $\frac{x^2}{9}-\frac{y^2}{(9/4)}=1$

** Foci $F(\frac{3}{2}\sqrt{5},0)$, $F^{/}(-\frac{3}{2}\sqrt{5},0)$

(as: $c^2=a^2+b^2 \to c^2=9+(9/4)=(45/4) \to c=\frac{3}{2}\sqrt{5}$)

** Asymptotes : $y=\pm\frac{b}{2}x \quad \Rightarrow \quad y=\pm\frac{1}{2}x$.

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<u>E x e r c i s e</u> (1–4)

<u>H y p e r b o l a</u>

I- Discus the graph of the following equations ,draw that graphs and deduce all of its available .information .

1) $4x^2-25y^2=100$, 2) $4x^2-9y^2-32x-36y-8=0$
3) $9x^2-25y^2=225$, 4) $4x^2-y^2+32x-8y+49=0$
5) $4y^2-16x^2-64=0$, 6) $25x^2-16y^2+250x+32y+109=0$
7) $25x^2-9y^2+100x-54y+10=0$, 8) $\frac{(y+3)^2}{49}-\frac{(x+2)^2}{16}=1$.
9) $9y^2-x^2+12x-36y-36=0$. , 10) $\frac{(x-3)^2}{9}-\frac{(y-4)^2}{36}=1$
11) $25x^2-9y^2+100x-54y+10=0$, 12) $\frac{(x+4)^2}{49}-\frac{y^2}{25}=1$

II- Deduce the standard equation of each of the following hyperbola that has the given information :

1) Center $O(0,0)$, vertices $V(0,\pm1)$ and foci $F(0,\pm4)$.

2) Center $O(0,0)$, vertices $V(\pm5,0)$ and foci $F(\pm8,0)$.

3) Center $O(0,0)$, vertices $V(\pm3,0)$ and pass through the point $P(8,2)$.

4) Center $O(0,0)$, vertices $(\pm3,0)$ and asymptotes $y=\pm2x$.

5) Center $O(0,0)$, foci $(0,\pm10)$ and asymptotes $y=\pm\frac{1}{3}x$.

6) Center $O(0,0)$, vertices $(\pm2,0)$ and foci $(\pm6,0)$.

7) Center $O(0,0)$, foci $(\pm5,0)$ and the distance between vertices $2a=8$.

8) Center $O(2,-4)$, one of its focus $(7,-4)$ and the distance between vertices $2a=8$.

§§§§§§§§§§§§

CHAPTER 2

PARAMETRIC EQUATIONS AND POLAR COORDINATES

I- PARAMETRIC EQUATIONS

I-1 Parametric Equations

When the path of a point moving in the plane looks like the curve in Fig. (2.1), we cannot hope to describe it with a Cartesian formula that expresses **y** directly in terms of **x** or **x** directly in terms of **y.** Instead, we express each points coordinates as a function of time **t**
and describe the path with a pair of equations

$$x = f(t)\,, \quad y = g(t).$$

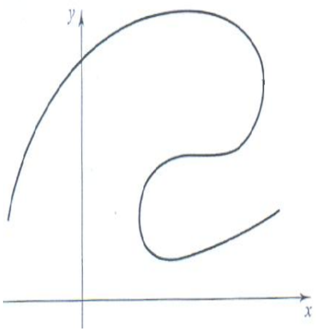

***Fig*. (2.1)**

***Definition* : 1**

A plane curve is a set **C** of ordered pairs $(f(t), g(t))$ where **f** and **g** are continuous functions on an interval **I**.

The graphs of several curves are sketched in Fig. (2.2), where **I** is a closed interval $[a, b]$. In (i) $P(a) \neq P(b)$, and P(a) and P(b) are called the **end points** of **C**. The curve in (i) intersects itself; that is, two different values of *t* produce the same point. If $P(a) = P(b)$, as in (ii), then **C** is **closed curve.** If $P(a) = P(b)$ and **C** does not intersect it self at any other point, as in (iii), then **C** is a **simple closed curve.**

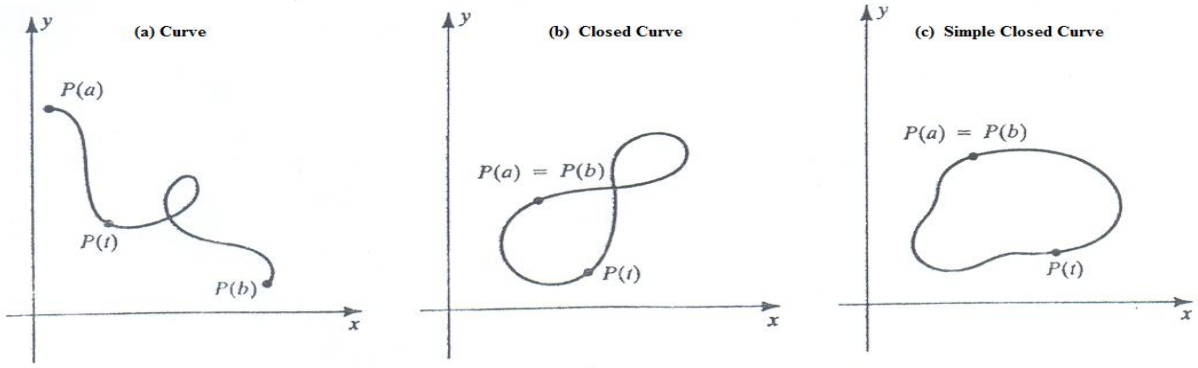

Fig. (2.2)

***Definition* : 2**

Let **C** be the curve consisting of all ordered pairs $(f(t), g(t))$ where **f** and **g** are continuous functions on an interval **I** . The equations $x = f(t)$, $y = g(t)$ for $t \in I$ are parametric equations for the curve **C** with parameter **t.**

***Example* :1**

Sketch the graph of the curve **C** that has the parameterization :

$$x = 2t, \qquad y = t^2 - 1; \qquad -1 \le t \le 2$$

Solution:

t	-1	-1/2	0	1/2	1	3/2	2
x	-2	-1	0	1	2	3	4
y	0	-3/4	-1	-3/4	0	5/4	3

The arrowheads on the graph indicate the direction in which $P(x, y)$ traces the curve **C** as **t** increases from -1 to 2. [see Fig. (2.3)].

By eliminating the parameter *t*, we obtain the equation in Cartesian form as,

$$y = \left(\frac{x}{2}\right)^2 - 1; \qquad -2 \le x \le 4.$$

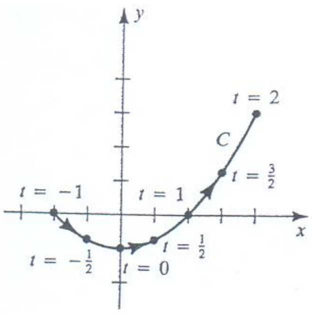

Fig. (2.3)

The graph of the curve **C** is that part of the parabola (symmetric about the *y-axis* with vertex at (0, 1)) between the points (-2, 0) and (4, 3). The orientation of the parameterized curve **C** is the direction determined by increasing values of the parameter. This orientation is indicated by arrowheads on **C**.

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Example :2

A point moves in a plane such that its position $P(x, y)$ at time **t** is given by:

$$x = a \cos t, \qquad y = a \sin t; \qquad t \in \Re,$$

where $a > 0$. Describe the motion of the point.

Solution :

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	a	$\frac{a}{\sqrt{2}}$	0	$\frac{-a}{\sqrt{2}}$	$-a$	$\frac{-a}{\sqrt{2}}$	0	$\frac{a}{\sqrt{2}}$	a
y	0	$\frac{a}{\sqrt{2}}$	a	$\frac{a}{\sqrt{2}}$	0	$\frac{-a}{\sqrt{2}}$	$-a$	$\frac{-a}{\sqrt{2}}$	0

[See Fig. (2.4)]. We may eliminate the parameter by rewriting the parametric equation as,

$$\frac{x}{a} = \cos t, \qquad \frac{y}{a} = \sin t,$$

and using the identity $\cos^2 t + \sin^2 t = 1$, to obtain, $x^2 + y^2 = a^2$, which is a circle C of radius a with center at the origin as shown.

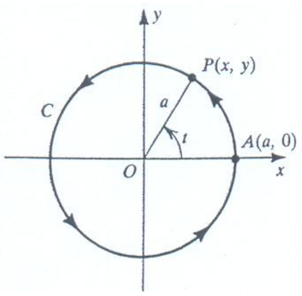

Fig. (2.4)

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Example :3

Sketch the graph of the curve C that has the parameterization:

$$x = -2 + t^2, \qquad y = 1 + 2t^2; \qquad t \in \Re$$

and indicate the orientation.

Solution :

t	-3	-2	-1	0	1	2	3
x	7	2	-1	-2	-1	2	3
y	19	9	3	1	3	9	19

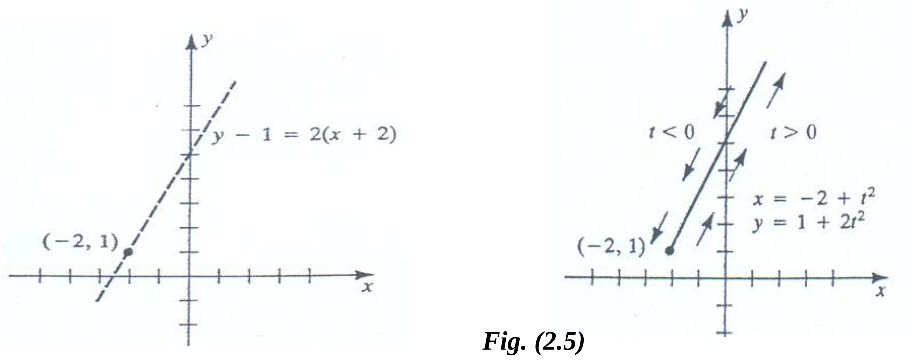

Fig. (2.5)

By eliminating the parameter *t*, we obtain the equation in Cartesian form as: $y = 2x + 5$

It is an equation of the line of slope 2 through the point (-2, 1) as shown.

Since $t^2 \geq 0$, thus the graph of **C** is that part of the line to the right of the point (-2, 1) (which corresponding to the value $t = 0$.)

The orientation is indicated by the arrows alongside of **C.** As **t** increases in the interval $(-\infty, 0]$, the point $P(x, y)$ moves down the curve toward the point (-2, 1).

As **t** increases in $[0, \infty)$, the point $P(x, y)$ moves up the curve away from the point (-2, 1).

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<u>Example</u> :4

Find three parameterizations for the line of slope *m* through the point (x_1, y_1)

<u>Solution :</u>

By the point-slope form, an equation for the line is: $y - y_1 = m(x - x_1)$.

let : $x = t$, then $y - y_1 = m(t - x_1)$, and we obtain the parameterization,

$x = t, \quad y = y_1 + m(t - x_1); \quad t \in \Re$.

We obtain another parameterization for the line if we let, $x - x_1 = t$.

In this case, $y - y_1 = mt$, and we obtain the parameterization,

$$x = x_1 + t, \qquad y = y_1 + mt; \qquad t \in \Re.$$

For third parameterization, let $x - x_1 = \tan t$, then $y - y_1 = m \tan t$, and we obtain the parameterization,

$$x = x_1 + \tan t, \qquad y = y_1 + m \tan t; \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

We can find many other parameterizations for the line .

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Example :5

A computer-generated graph of the figure :

$$x = \sin 2t, \quad y = \cos t\,; \quad 0 \le t \le 2\pi$$

is shown in Fig. (2.6), with the arrowheads indicating the orientation. Verify the orientation , and find an equation in *x* and *y* for the curve.

Solution:

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x	0	1	0	-1	0	1	0	-1	0
y	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1

As **t** increases from 0 to $\pi/2$, the point $P(x, y)$ starts at (0,1) and traces the part of the curve in quadrant **I** in clockwise direction. As t increases from $\pi/2$ to π, the point $P(x, y)$ traces the part in quadrant **III** in a counterclockwise direction.

For $\pi \le t \le 3\pi/2$, we obtain the part in quadrant **IV**, and $3\pi/2 \le t \le 2\pi$ gives us the part in quadrant **II**.

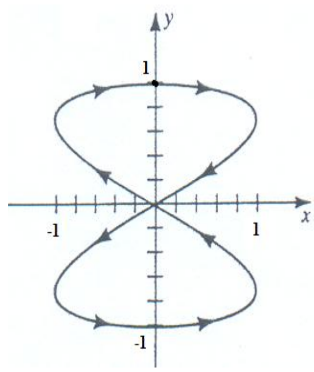

Fig. (2.6)

Now, $\quad x = \sin 2t = 2\sin t \cos t$.

Then,

$$x^2 = 4\sin^2 t \cos^2 t = 4\,(1 - \cos^2 t)\cos^2 t = 4\,(1 - y^2)\,y^2$$

or

$$4y^4 - 4y^2 + x^2 = 0$$

Solving for y : $\quad y^2 = \dfrac{4 \pm \sqrt{16 - 16x^2}}{8} = \dfrac{1 \pm \sqrt{1 - x^2}}{2}$

$$y = \sqrt{\frac{1 \pm \sqrt{1 - x^2}}{2}}\,.$$

These complicated equations should indicate the advantage of expressing the curve in parametric form.

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Example :6

The curve traced by a fixed point P on the circumference of a circle as the circle rolls along a line in a plane is called a **Cycloid**. Find parametric equations for a cycloid and determine the intervals on which it is smooth.

Solution :

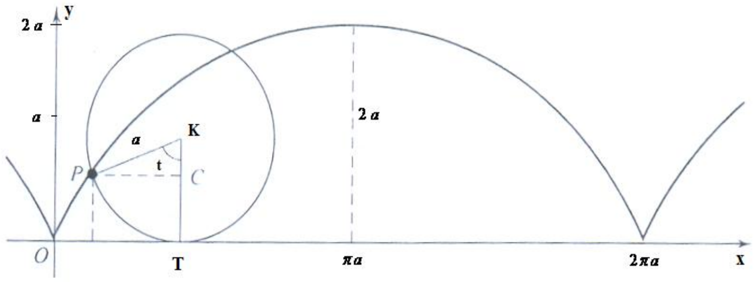

Fig. (2.7)

Let K denotes the center of the circle and T the point of tangency with the **x-axis**. Let **t** be the radian angle TKP. Thus the distance from O to T is $d(O,T) = a\,t$

K is $(a\,t,\ a)$. Translate the axes to $K(a\,t,\ a)$, then

$$x = at + x', \qquad y = a + y', \qquad \theta = \frac{3\pi}{2} - t\,.$$

Since $\quad \theta + t + \frac{\pi}{2} = 2\pi,\quad$ from the graph, we get:

$$x' = a\cos\theta = a\cos\left(\frac{3\pi}{2} - t\right) = -a\sin t$$

$$y' = a\sin\theta = a\sin\left(\frac{3\pi}{2} - t\right) = -a\cos t\,,$$

then, $\quad x = a(t - \sin t), \qquad y = a(1 - \cos t); \qquad t \in \Re$

$$\frac{dx}{dt} = a(1 - \cos t), \qquad \frac{dy}{dt} = a\sin t$$

these derivatives are continuous for every *t*, but are simultaneously 0 at $t = 2n\pi$ for every integer *n*. Then the cycloid is smooth in the interval $[2n\pi,\ 2(n+1)\pi]$ for every integer *n*.

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Example ***:7***

Sketch the graph of the curve *C* that has the parameterization:

$$x = \cos^3 t, \quad y = \sin^3 t\ , \quad 0 \le t \le 2\pi$$

This curve is called **the Asteroid**

Solution:

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
X	1	$\frac{1}{2\sqrt{2}}$	*0*	$\frac{-1}{2\sqrt{2}}$	-1	$-1/2\sqrt{2}$	*0*	$\frac{1}{2\sqrt{2}}$	1
Y	*0*	$\frac{1}{2\sqrt{2}}$	1	$\frac{1}{2\sqrt{2}}$	*0*	$\frac{-1}{2\sqrt{2}}$	-1	$\frac{-1}{2\sqrt{2}}$	*0*

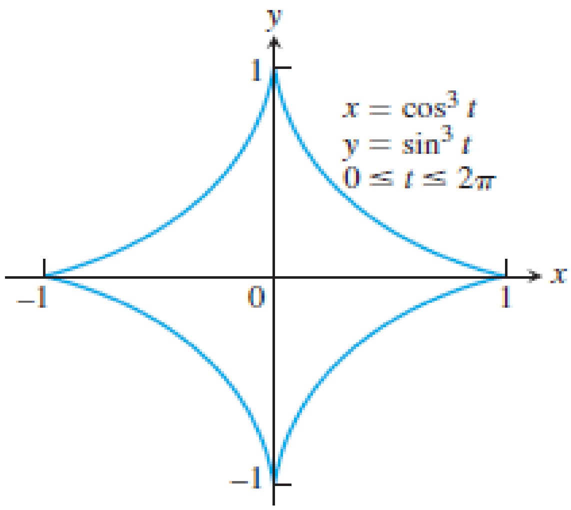

Fig. (2.8)

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<u>**Exercise**</u> **(2 – 1)**

(I) Find an equation in *x* and *y* whose graph contains the points on the curve *C*. Sketch the graph of *C*.

(1) $x = t - 2, \quad y = 2t + 3; \quad 0 \le t \le 5.$

(2) $x = 1 - 2t, \quad y = t + 1; \quad -1 \le t \le 4.$

(3) $x = e^{t}, \quad y = e^{-2t}; \quad t \in \Re.$

(4) $x = e^{t}, \quad y = e^{-t}; \quad t \in \Re.$

(5) $x = \sec t, \quad y = \tan t; \quad -\pi/2 \le t \le \pi/2.$

(6) $x = \cos 2t, \quad y = \sin t; \quad -\pi \le t \le \pi.$

(7) $x = 3\cosh t, \quad y = 2\sinh t; \quad t \in \Re.$

(8) $x = \cosh t, \quad y = \sinh t; \quad t \in \Re.$

(9) $x = (t + 1)^3, \quad y = (t + 2)^2; \quad 0 \le t \le 2.$

(10) $x = \cos t, \quad y = \sin t; \quad 0 \le t \le \pi.$

(II) Show that : $x = a\cos t + h, \; y = b\sin t + h; \; 0 \le t \le 2\pi$ are parametric equations of an ellipse with center (*h*, *k*), and axes of lengths $2a$ and $2b$.

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I-2 Derivatives , Arc Length And Surface Area

From the previous course "Math A", If a curve is described by an equation $y = f(x)$, where *f* is a differentiable function, we know how to find the slope of a tangent line at a point on the curve, the length of the curve, and the area of the surface of revolution obtained by revolving the curve about an axis. In this section, we discuss how to find these quantities when the curve is described by parametric equations.

***Theorem* : 1**

If a smooth curve **C** is given parametrically by $x = f(t)$, $y = g(t)$, then the slope of the tangent line to *C* at $P(x, y)$ is : $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ provided $\frac{dx}{dt} \neq 0$.

***Example* :8**

Find the equation of tangent to the curve,, $x = \sec t$, $y = \tan t$; $\frac{-\pi}{2} < t < \frac{\pi}{2}$, at the point $\left(\sqrt{2}, 1\right)$, where $t = \pi/4$

Solution :

The slope of the curve at *t* is , $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$,

at $t = \pi/4$, $\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{\sec(\pi/4)}{\tan t\ (\pi/4)} = \frac{\sqrt{2}}{1} = \sqrt{2}$.

The equation of tangent is: $y - 1 = \sqrt{2}\,(x - \sqrt{2})$ or $y = \sqrt{2}\,x - 1$

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***Example* :9**

Let **C** be the curve with parameterization:

$$x = 2t, \quad y = t^2 - 1; \quad -1 \le t \le 2.$$

Find the equations of the tangent and normal lines to **C** at $t = 1$.

Solution :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t$$

The slope of the tangent line to **C** at $t = 1, m_1 = 1$, and the slope of the normal line to **C** at $t = 1$, $m_2 = -1$. The point corresponding to $t = 1$ is $P(2, 0)$

equation of tangent line : $y = x - 2$ and equation of normal line : $y = -x + 2$.

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Example :10

Let **C** be the curve with parameterization,

$$x = t^3 - 3t, \quad y = t^2 - 5t - 1; \quad t \in \Re .$$

a) Find an equation of the tangent line to **C** at the point corresponding to $t = 2$.

b) For what values of **t** is the tangent line horizontal or vertical **?**

Solution :

a) Using the parametric equations for **C**, we find that the point corresponding to $t = 2$ is: $P(2, -7)$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-5}{3t^2-3}$

the slope **m** of the tangent line at (2,-7) is : $m = \left.\frac{2t-5}{3t^2-3}\right)_{t=2} = -\frac{1}{9}$

The equation of the tangent line is :

$$y + 7 = -\frac{1}{9}(x-2) \quad \text{or} \quad x + 9y + 61 = 0.$$

b) The tangent is **horizontal** if : $\frac{dy}{dx} = \frac{2t-5}{3t^2-3} = 0$, i.e. $2t - 5 = 0$, or $t = \frac{5}{2}$.

The corresponding point on **C** is $\left(\frac{65}{8}, \frac{29}{4}\right)$.

The tangent is **vertical** if: $\frac{dy}{dx} = \frac{2t-5}{3t^2-3} = \infty$, i.e. $3t^3 - 3 = 0$, or $t = \pm 1$.

The corresponding points on *C* are $(-2, -5), (2, 5)$.

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Theorem : 2

If a smooth curve **C** is given parametrically by $x = f(t)$, $y = g(t)$, and if y' is differentiable function of t then the second derivative in parametric form,

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} \quad \text{provided} \quad \frac{dx}{dt} \neq 0.$$

N O T E

$$\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{dx^2/dt^2}$$

Example :11

Find d^2y/dx^2 as a function of **t** if $x = t - t^2$, $y = t - t^3$; $t \in \Re$

Solution :

As : $y' = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 3t^2}{1 - 2t}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2 - 6t + 6t^2}{(1 - 2t)^2} \frac{1}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3}$$

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Example :12

Let C be the curve with parameterization, $x = e^{-t}$, $y = e^{2t}$; $t \in \Re$

a) Sketch the graph of C and indicate the orientation.

b) Find $\dfrac{d^2y}{dx^2}$.

c) Find a function K that has the same graph as C, and use $K'(x)$ and $K''(x)$ to check the answers to (b)

d) Discuss the concavity of C.

Solution:

a) To get the graph of C, eliminate the parameter,

$x = e^{-t} = 1/e^t$, *i.e.* $e^t = \dfrac{1}{x}$, then $y = \left(\dfrac{1}{x}\right)^2 = \dfrac{1}{x^2}$.

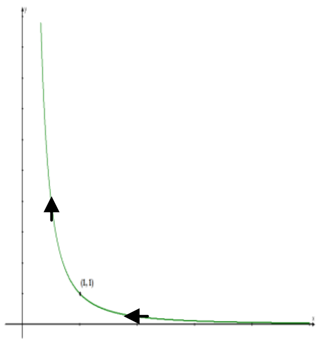

Fig. (2.9)

Note that, $x = e^{-t} > 0$, $y = e^{2t} > 0$.

The point (1,1) corresponds to $t = 0$. If t increases in $(-\infty, 0]$, the point $P(x, y)$ approaches (1, 1) from the right. If t increases in $[0, \infty)$, the point $P(x, y)$ moves up the curve approaching the *y-axis*.

b) $y' = \dfrac{dy/dt}{dx/dt} = \dfrac{2e^{2t}}{-e^{-t}} = -2e^{3t} \Rightarrow y'' = \dfrac{dy'/dt}{dx/dt} = \dfrac{-6e^{3t}}{-e^{-t}} = 6e^{4t}$

c) From (a), a function K that has the same graph as C is given by

$K(x) = \dfrac{1}{x^2} = x^{-2}$; $x > 0$. $\Rightarrow$ $K'(x) = -2x^{-3} = -2(e^{-t})^{-3} = -2e^{3t}$.

$\Rightarrow$ $K''(x) = 6x^{-4} = 6(e^{-t})^{-4} = 6e^{4t}$. This values agree with the results in (b).

d) Since $\dfrac{d^2y}{dx^2} = K''(x) = 6e^{4t} > 0$ $\forall t \in \Re$, the curve C is concave upward at every point.

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Example :13

Find the area enclosed by the Asteroid: $x = \cos^3 t, \quad y = \sin^3 t\ , \quad 0 \le t \le 2\pi$ [see fig 2.8]

Solution :

By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0 \le t \le \pi/2$. We can apply the definite integral formula for area studied in Math (1), using substitution to express the curve and differential *dx* in terms of the parameter *t*. So,

$$\text{Area} = 4\int_0^1 y\,dx = 4\int_{\pi/2}^{0} \sin^3 t * 3\cos^2 t\,[-\sin t]\,dt$$

$$= 12\int_0^{\pi/2} \sin^4 t \cos^2 t\,dt = 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \left(1 - \cos 2t - \cos^2 2t + \cos^3 2t\right) dt = \frac{3}{8}\pi$$

If a curve **C** is the graph of $y = f(x)$ and the function **f** is smooth on $[a, b]$, then the length of *C* is given by : $L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$; The next theorem give a formula for finding length of parameterized curve.

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***Theorem :* 3**

If a smooth curve ***C*** is given parametrically by $x = f(t),\ y = g(t),\ a \le t \le b$, and if **C** does not intersect itself, except possibly for $t = a$ and $t = b$, then **the length** L of **C** is : $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

The integral formula in theorem (3) is not necessarily true if **C** intersects itself.

<u>Example</u> :14

Find the length of one arch of the cycloid that has the parameterization,

$$x = t - \sin t, \quad y = 1 - \cos t; \qquad t \in \Re$$

<u>Solution:</u>

The graph has the shape as shown in Fig (2.10) , the radius a of the circle is 2.

One arch is obtained if *t* varies from 0 to 2π.

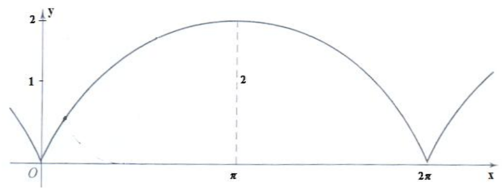

Fig. (2.10)

$$L = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} \ dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \ dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t} \ dt \quad = \int_0^{2\pi} \sqrt{2} \ \sqrt{1 - \cos t} \ dt$$

but $\sin^2(t/2) = (1 - \cos t)/2$, then

$$L = \int_0^{2\pi} \sqrt{2} \ \sqrt{2\sin^2(t/2)} \ dt = \int_0^{2\pi} 2 \sin(t/2) \ dt = -4\left(\cos(t/2)\right)_0^{2\pi} = 8.$$

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<u>Example</u> :15

Find the length in the first quadrant of the Asteroid : $x = \cos^3 t, \quad y = \sin^3 t$

<u>Solution:</u>

$$\left(\frac{dx}{dt}\right)^2 = (-3\cos^2 t \sin t)^2 = 9\cos^4 t \sin^2 t,$$

$$\left(\frac{dy}{dt}\right)^2 = (3\sin^2 t \cos t)^2 = 9\sin^4 t \cos^2 t$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$$

$$= \int_0^{\pi/2} \sqrt{9\sin^2 t \cos^2 t \ (\sin^2 t + \cos^2 t)} \ dt$$

$$= \int_0^{\pi/4} 3\sin t \cos t \ dt = \left.\frac{3}{2}\sin^2 t\right)_0^{\pi/2} = \frac{3}{2}(1 - 0) = \frac{3}{2} = 1.5$$

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***Theorem* : 4**

If a smooth curve **C** is given parametrically by $x = f(t)$, $y = g(t)$, $a \le t \le b$, and if **C** does not intersect itself, except possibly for $t = a$ and $t = b$, then the **area S of the surface** of revolution obtained by revolving C is

$$S = \int_a^b 2\pi y\, dL = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad ; \text{ about the } \textbf{x-axis}$$

$$S = \int_a^b 2\pi x\, dL = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad ; \text{ about the } \textbf{y-axis}$$

***Example* :16**

Find the area of the surface generated by revolving the curve:

$x = \cos t, \quad y = 1 + \sin t; \quad 0 \le t \le 2\pi$; about the **x-axis**

Solution:

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 2\pi \int_0^{2\pi} (1+\sin t)\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt = 2\pi \left(t - \cos t\right)_0^{2\pi} = 4\pi^2$$

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***Example* :17**

Verify that the surface area of a sphere of radius a is $4\pi a^2$.

Solution :

If C is the upper half of the circle: $x^2 + y^2 = a^2$,
then the spherical surface may be obtained by revolving C about the *x-axis* Fig. (2.11).
Parametric equations for C are:

$$x = a\cos t, \quad y = a\sin t; \quad 0 \le t \le \pi$$

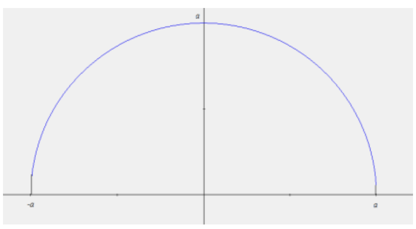

Fig. (2.11)

$$S = \int_a^b 2\pi y\, dL = 2\pi \int_0^{\pi} a\sin t \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$= 2\pi a^2 \int_0^{\pi} \sin t\, dt = 2\pi a^2 \left(-\cos t\right)_0^{\pi} = 2\pi a^2 (-1-1) = 4\pi a^2.$$

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Exercise (2 – 2)

(I) Find the slopes of the tangent line and the normal line at the point on the curve that corresponds to t = 2.

(1) $x = 4t^2 - 5, \quad y = t^2 - 1; \quad -2 \le t \le 2.$

(2) $x = t^3 + 1, \quad y = t^3 - 1; \quad -2 \le t \le 2.$

(3) $x = 4t^2 - 5, \quad y = 2t + 3; \quad t \in \Re.$

(4) $x = t^3, \quad y = t^2; \quad t \in \Re.$

(5) $x = \sqrt{t}, \quad y = 4t + 3; \quad t \ge 0.$

(6) $x = 2\sin t, \quad y = 3\cos t; \quad 0 \le t \le 2\pi.$

(7) $x = \cos t - 2, \quad y = \sin t + 3; \quad 0 \le t \le 2\pi.$

(II) Find the points on the curve *C* at which the tangent line is either horizontal or vertical.

Find $\frac{d^2y}{dx^2}$.

(1) $x = 4t^2, \quad y = t^3 - 12t; \quad t \in \Re$

(2) $x = t^3 - 4t, \quad y = t^2 - 4; \quad t \in \Re$

(3) $x = 3t^2 - 6t, \quad y = \sqrt{t}; \quad t \ge 0.$

(4) $x = \cos 2t, \quad y = \sin^2 t; \quad 0 \le t \le \pi$

(5) $x = \cosh t, \quad y = \sinh t; \quad t \in \Re.$

(6) $x = \cos^3 t, \quad y = \sin^3 t; \quad 0 \le t \le 2\pi$

(III) Find the length of the curve,

(1) $x = 5t^2, \quad y = 2t^3; \quad 0 \le t \le 1.$

(2) $x = 3t, \quad y = 2t^{3/2}; \quad 1 \le t \le 4$

(3) $x = e^t \cos t, \quad y = e^t \sin t; \quad 0 \le t \le \pi/2.$

(4) $x = \cos 2t, \quad y = \sin^2 t; \quad 0 \le t \le \pi$

(5) $x = \cos^3 t, \quad y = \sin^3 t; \quad 0 \le t \le \pi/2$

(IV) Find the area of the surface generated by revolving of the curve *C* about the *x-axis*,

(1) $x = t^2, \quad y = 2t; \quad 1 \le t \le 4.$

(2) $x = t^2, \quad y = t - \frac{1}{3}t^3; \quad 0 \le t \le 1.$

(3) $x = 4t, \quad y = t^3; \quad 1 \le t \le 2.$

(4) $x = t - \sin t, \quad y = 1 - \cos t; \quad 0 \le t \le 2\pi$

(V) Find the area of the surface generated by revolving of the curve *C* about the *y-axis*,

(1) $x = 4\sqrt{t}, \quad y = \frac{1}{2}t^2 + \frac{1}{t}; \quad 1 \le t \le 4.$

(2) $x = 3t, \quad y = t + 1; \quad 0 \le t \le 5.$

(3) $x = e^t \cos t, \quad y = e^t \sin t; \quad 0 \le t \le \pi/2.$

(4) $x = 3t^2, \quad y = 2t^3; \quad 0 \le t \le 1$

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II- POLAR COORDINATES

II-1 Polar And Cartesian Coordinates

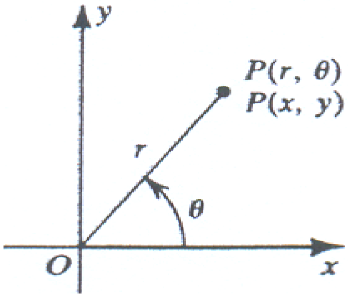

Fig. (2.12)

In a rectangular coordinate system, the order pair (a, b) denotes the point whose directed distances from the *x-axis* and *y-axis* are b and a respectively. Another method for representing points is to use **polar coordinates**.
We begin with a fixed point 0 (the origin, or pole) and a directed line ***(the polar axis)*** with end point 0.

Next we consider any point *P* in the plane different from 0. If , as illustrated in Fig (2. 12) $r = d(0, p)$ and θ denotes the measure of any angle determined by the polar axis and OP, then *r* and θ are polar coordinates of *P*, the polar coordinates of a point are not unique. For example, the points $P(r, \theta)$, $P(r, \theta \pm 2n\pi)$; $n = 1, 2, 3, \ldots$
We agree that the pole **0** has polar coordinates $(0, \theta)$ for any θ.

Any point $P(r, \theta)$ in the polar coordinate is denoted by $P(x, y)$ in the rectangular coordinate system as illustrated in Fig. (2.12), so the question

" What is the relation between the polar coordinate and the rectangular coordinate systems ? "

The question is arise now and the answer in the following theorem.

Theorem : 5

The rectangular coordinates of the point $P(x, y)$ and the polar coordinates $P(r, \theta)$ are related as follows:

(i) $x = r\cos\theta, \quad y = r\sin\theta$; (ii) $r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

***Example* :18**

Find the polar equation for the circle : $x^2 + y^2 = 9$,

Solution:

Substituting $x = r\cos\theta,\ y = r\sin\theta$ to the given equation we obtain the corresponding polar equation $r = 3$ which is a circle centered at origin with radius 3.

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***Example* :19**

Find a polar equation for the circle : $x^2 + (y-3)^2 = 9$,

Solution :

Substituting $x = r\cos\theta,\ y = r\sin\theta$ to the given equa

$\Rightarrow \quad r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9,$

$\Rightarrow \quad r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$

$\Rightarrow \quad r^2 - 6r\sin\theta = 0, \quad r = 0 \text{ or } r = 6\sin\theta$

which is a circle centered at (0, 3) with radius 3.

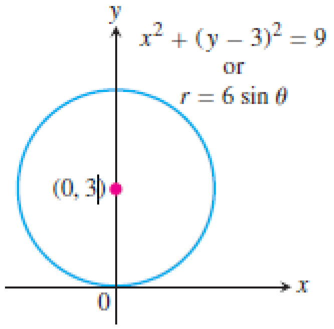

Fig (2.13)

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***Example* :20**

Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(i) $r\cos\theta = 5$, (ii) $r^2 = 4r\cos\theta$ (iii) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution:

Use the substitution $x = r\cos\theta,\ y = r\sin\theta$

(i) $r\cos\theta = 5$ i.e. $x = 5$

The graph is **vertical line** through $x = 5$.

(ii) $r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$ i.e $(x-2)^2 + y^2 = 4$

The graph is **circle** , radius 2 and centered at (2, 0)

(iii) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

$2r\cos\theta - r\sin\theta = 4$ i.e. $2x - y = 4$ or $y = 2x - 4$

The graph is **line** with slope $m = 2$ and y-intercept $b = -4$

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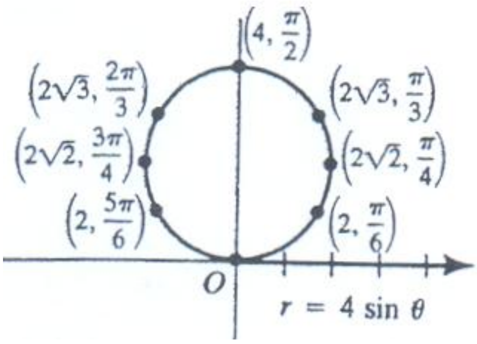
Example :21

Sketch the graph of the polar equation $r = 4\sin\theta,\ 0 \le \theta \le \pi$.

Solution:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	2	$2\sqrt{2}$	$2\sqrt{3}$	4	$2\sqrt{3}$	$2\sqrt{2}$	2	0



If θ vary from π to 2π, the obtained points are the same as obtained above. It is a circle of center at $(2, \pi/2)$ with radius 2,

In general, by using the same method as in the preceding example, we can show that the graph $r = a\sin\theta$, with $a \ne 0$, is a circle of radius of radius $a/2$ of the type illustrated in Fig. (2.14a), and the graph: $r = a\cos\theta$, with $a \ne 0$, is a circle of radius of radius $a/2$ of the type illustrated in Fig. (2.14b), and the graph

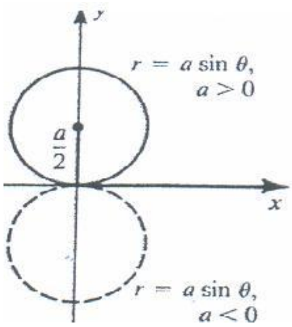


Fig. (2.14a)

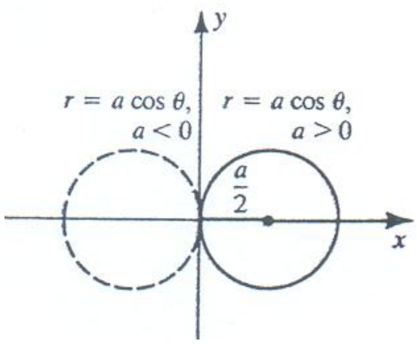


Fig. (2.14b)

**** A Cardioid Or a Heart – Shaped ****

Example :22

Sketch the graph of the polar equation : $r = 2 + 2\cos\theta$

Solution:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0

Since the cosine function decreases from 1 to -1 as θ varies from 0 to π , it follows that r decreases from 4 to 0 in this θ interval. Plotting these points in the $r\theta$ plane leads to the upper half of the graph sketched in Fig. (2.15). If θ increases from π to 2π , then the cosine function increases from -1 to 1 and r increases from 0 to 4. Plotting points for $\pi \le \theta \le 2\pi$ gives us the lower half of the graph.

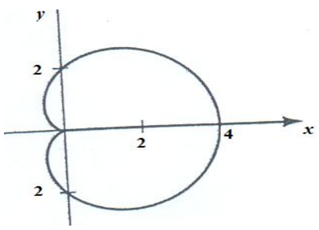


Fig. (2.15b)

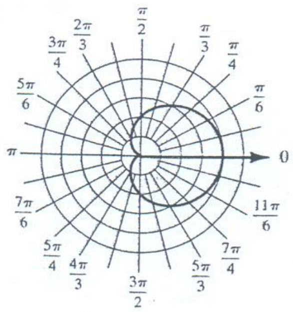


Fig. (2.15a)

Plotting points corresponding to $0 \le \theta \le \pi$, in Fig. (2.16), the graph of any of the following polar equations, with $a \neq 0$, is a cardioid .

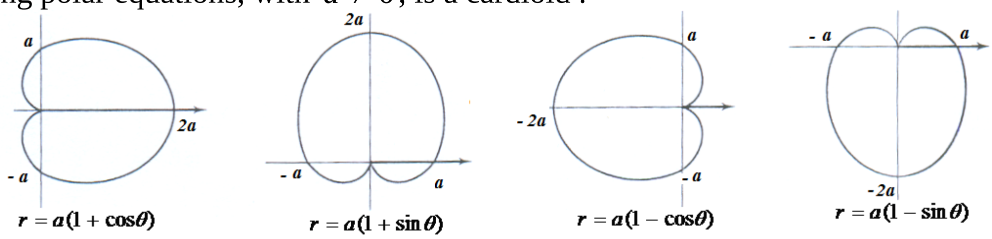


Fig. (2.16)

If the polar equation is of the form: $r = a + b\cos\theta$ or $r = a + b\sin\theta$

the graphs are called **limacons**.

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The special case of **limacons** in which $|a| = |b|$ are **cardioids.** Some limacons contain a loop, as the following example.

Example :23

Sketch the graph of the polar equation: $r = 2 + 4\cos\theta$

Solution

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	6	$2+2\sqrt{3}$	$2+2\sqrt{2}$	4	2	0	$2-2\sqrt{2}$	$2-2\sqrt{3}$	-2

In the graph in Fig. (2.17), θ varies from 0 to π gives us the lower half of the small loop and the upper part of the large loop, and from π to 2π gives us the rest of the graph.

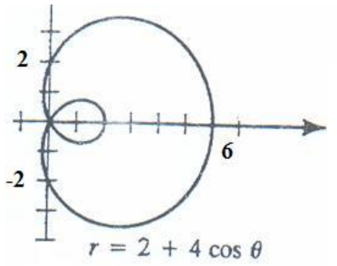

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Fig. (2.17)

Example :24

Sketch the graph of the polar equation: $r = \frac{3}{2} + \cos\theta$

Solution

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	5/2	$\frac{3+\sqrt{3}}{2}$	$\frac{3+\sqrt{2}}{2}$	2	3/2	1	$\frac{3-\sqrt{2}}{2}$	$\frac{3-\sqrt{3}}{2}$	1/2

In the graph in Fig. (2.18), θ varies from 0 to π gives us the upper half of the graph, and from π to 2π gives us the lower half of the graph.

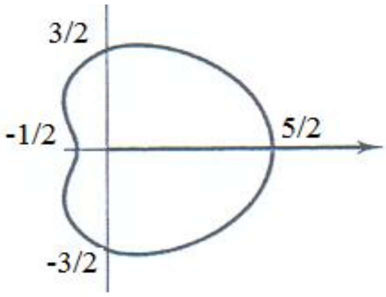

Fig. (2.18)

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** **A Four-Leafed Rose** **

Example :25

Sketch the graph of the polar equation: $r = a\sin 2\theta$ for $a > 0$

Solution

Instead of tabulating solutions. If θ increases from 0 to $\pi/4$, then 2θ increase from 0 to $\pi/2$ and Fig. (2.19) hence $\sin 2\theta$ increases from 0 to 2. It follows that r increases from 0 to a in the interval $[0, \pi/4]$.

If we next let θ increases from $\pi/4$ to $\pi/2$, then 2θ changes from $\pi/2$ to π and hence r decreases from a to 0 in the interval $[\pi/4, \pi/2]$. This gives us the graph in the 1st quadrant, the 2nd, 3rd, and 4th are the same. This graph is called **a four-leafed rose**. In general, a polar equation of the form,

$r = a\ \sin n\theta$ or $r = a\ \cos n\theta$

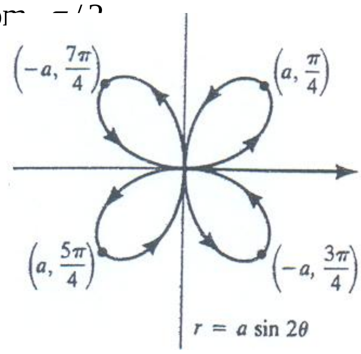


Fig. (2.19)

For any positive integer *n* greater than 1 and any non-zero real number a has a graph that consists of a number of loops through the origin.

If *n* is even, there are 2*n* loops and if *n* is odd, there are *n* loops .

Different cases are illustrated in Fig. (2.20a) [Lemniscates], (2.20b) [Three leaved rose], and (2.20c) [8 leaved rose]

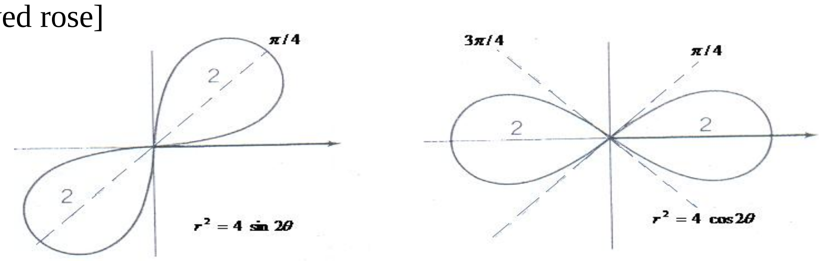
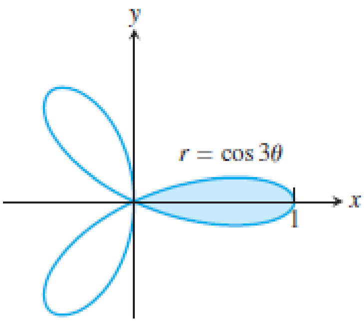


Fig. (2.20a)



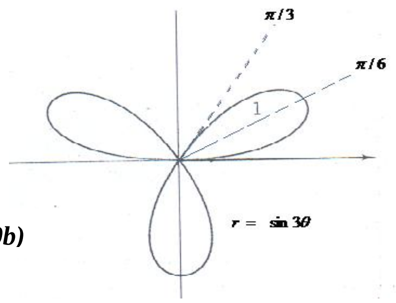


Fig. (2.20b)

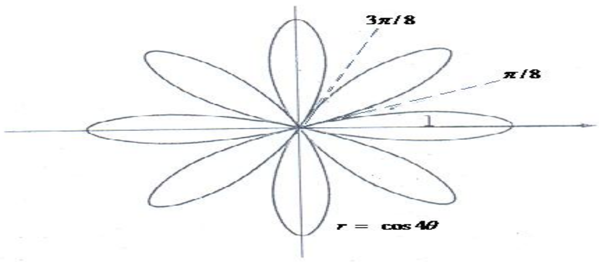


Fig. (2.20c)

Example :26

Sketch the graph of the polar equation $r = \theta$ *for* $\theta \geq 0$.

Solution

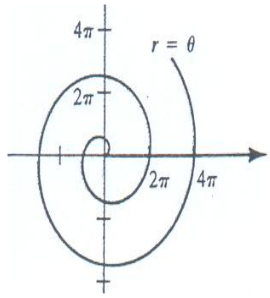


Fig. (2.21)

The graph consists of all points that have polar coordinates of the form (c, c) for any real number $c \geq 0$. Thus the graph contains the points $(0, 0)$, $(\pi/2, \pi/2)$, (π, π), and so on.

As θ increase r increase at the same rate, and the spiral winds around the origin in a counterclockwise direction, intersecting the polar axis at $0, 2\pi, 4\pi, \ldots,$ as illustrated.

In general, The graph of the polar equation $r = a\,\theta$ *for* any non zero real number a is a **Spiral of Archimedes.**

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Example :27

Find the polar equation for the hyperbola $x^2 - y^2 = 16$

Solution

Substituting, $x = r\cos\theta$, $y = r\sin\theta$

$$r^2(\cos^2\theta - \sin^2\theta) = 16 \qquad \text{or} \qquad r^2\cos 2\theta = 16$$

$$r^2 = \frac{16}{\cos 2\theta} = 16\sec 2\theta$$

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Example :28

Find the polar equation of an arbitrary line.

Solution

The general equation of an arbitrary line is: $ax + by = c$

Substituting, $x = r\cos\theta$, $y = r\sin\theta$

$$a r\cos\theta + b r\sin\theta = c \qquad \text{or} \qquad r\,(a\cos\theta + b\sin\theta) = c$$

Then the polar equation of an arbitrary line is:

$$r = \frac{c}{a\cos\theta + b\sin\theta}\ .$$

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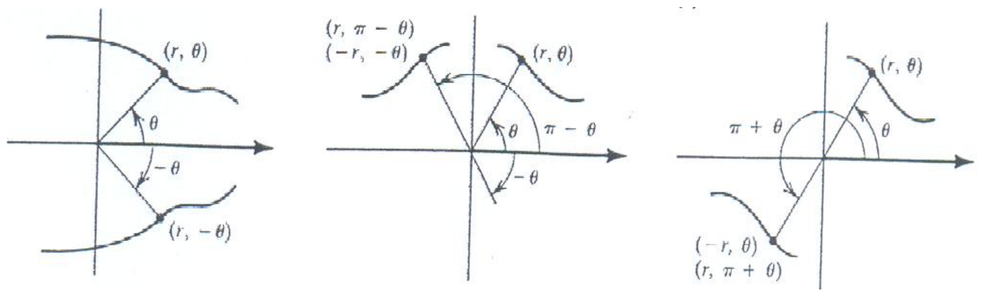
II-2 Slope Of Tangent Line In Polar Form

In x-y plane, the graph of $y = f(x)$ may be symmetric with respect to the *x-axis*, the *y-axis*, or the origin. So in the $r - \theta$ plane, the graph of $r = f(\theta)$ may be symmetric with respect to the *polar-axis*, the *line* $\theta = \pi/2$, or the pole.

Some typical symmetries are illustrated in Fig. (2.22)



(i) about polar-axis (ii) about line $\theta = \pi/2$ (iii) about pole

Fig. (2.22)

<u>This leads to the next RESULTS</u>

(1) The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if $f(-\theta) = f(\theta)$.

(2) The graph of $r = f(\theta)$ is symmetric with respect to the vertical line $\theta = \pi/2$ if

<u>either :</u>

a) $f(\theta) = f(\pi - \theta) \;\; \forall\theta$ **or** b) $f(-r, -\theta) = f(r, \theta)$

(3) The graph of $r = f(\theta)$ is symmetric with respect to the pole if either:

a) r can be replaced by $-r$ **or** b) $f(\theta) = f(\pi + \theta) \;\; \forall\theta$

Tangent lines to graphs of polar equations may be found by means of the next theorem.

<u>Theorem</u> : 6

The slope m of the tangent line to the graph of $r = f(\theta)$ at the point $P(r, \theta)$ is

$$m = \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right) \Big/ \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)$$

Proof

If (x, y) are the rectangular coordinates of $P(r, \theta)$ then,

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta,$$

$$m = \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \big(f(\theta)\cos\theta + f'(\theta)\sin\theta\big)/\big(f(\theta)(-\sin\theta) + f'(\theta)\cos\theta\big)$$

$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$$

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***Example* :29**

For the Cardioid $r = 2 + 2\cos\theta$ with $0 \le \theta \le 2\pi$, find,

(a) the slope of the tangent line at $\theta = \pi/6$.

(b) the points at which the tangent is horizontal or vertical.

Solution

$$m = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{(-2\sin\theta)\sin\theta + (2 + 2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (2 + 2\cos\theta)\sin\theta}$$

$$= \frac{2(\cos^2\theta - \sin^2\theta) + 2\cos\theta}{-2(2\sin\theta\cos\theta) - 2\sin\theta} = -\frac{\cos 2\theta + \cos\theta}{\sin 2\theta + \sin\theta}.$$

(a) **For** $\theta = \pi/6$,

$$m = -\frac{\cos 2\theta + \cos\theta}{\sin 2\theta + \sin\theta} = -\frac{\cos(\pi/3) + \cos(\pi/6)}{\sin(\pi/3) + \sin(\pi/6)}$$

$$= -\frac{(1/2) + (\sqrt{3}/2)}{(\sqrt{3}/2) + (1/2)} = -1$$

(b) To find horizontal tangents, $\cos 2\theta + \cos\theta = 0$, then

$2\cos^2\theta - 1 + \cos\theta = 0$ or $(2\cos\theta - 1)(\cos\theta + 1) = 0$

which gives, $\cos\theta = 1/2$ or $\cos\theta = -1$, *i.e.*

$$\theta = \pi/3, \quad 5\pi/3 \quad \text{or} \quad \theta = \pi.$$

The corresponding points at which the tangent is horizontal, $(3, \pi/3)$, $(3, 5\pi/3)$ and $(0, \pi)$

For the vertical tangent, $\sin 2\theta + \sin\theta = 0$

$2\sin\theta\cos\theta + \sin\theta = 0$ or $\sin\theta\,(2\cos\theta + 1) = 0$

i.e. $\sin\theta = 0$ or $\cos\theta = -1/2$, then $\theta = 0, \ \pi$ or $\theta = 2\pi/3, \ 4\pi/3$.

Since we found above that $\theta = \pi$ gives us a horizontal tangent line, then the points at which the tangent is vertical are $(4, 0)$, $(1, 2\pi/3)$ and $(1, 4\pi/3)$.

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Exercise (2 – 3)

(I) Sketch the graph of the polar equations,

(1) $r = 5$ (2) $\theta = \pi/4$ (3) $r = 3\cos\theta$
(4) $r = 1 + 2\cos\theta$ (5) $r = 4 + 4\sin\theta$ (6) $r = 2\sin 4\theta$
(7) $r^2 = 4\cos 2\theta$ (8) $r = 1 - \csc\theta$

(II) Find a polar equation that has the same graph as the equation in *x* and *y*.

(1) $x^2 + y^2 = 16$ (2) $2y = -x$ (3) $y^2 - x^2 = 4$
(4) $x\,y = 8$ (5) $r^2\cos^2\theta = 8\,r\sin\theta$ (6) $r\sin\theta = 6\,r\cos\theta$

(III) Find an equation in *x* and *y* that has the same graph as the polar equation and sketch the graph in $x - y$ plane

(1) $r\cos\theta = 5$ (2) $r\sin\theta = -2$ (3) $r\sin\theta - 2\,r\cos\theta = 6$
(4) $r = 4\sec\theta$ (5) $r\sin\theta + r^2\cos^2\theta = 1$ (6) $r^2\sin 2\theta = 4$

(IV) If *a* and *b* are non-zero real numbers, prove that the graph of $r = a\sin\theta + b\cos\theta$ is a circle, and find its center and radius.

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II-3 Integrals In Polar Coordinates

***Theorem* : 7**

If f is continuous and $f(\theta) \geq 0$ on $[\alpha, \beta]$, where $0 \leq \alpha < \beta \leq 2\pi$, then the area *A* of the region bounded by the graphs Fig. (2.23) of $r = f(\theta)$, $\theta = \alpha$ $\theta = \beta$ is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f^2(\theta) \, d\theta .$$

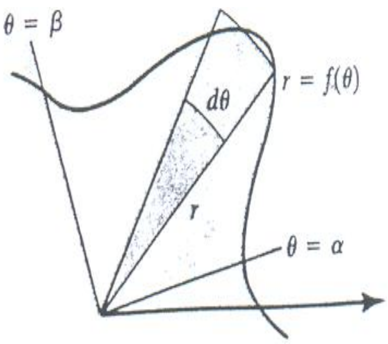

Fig. (2.23)

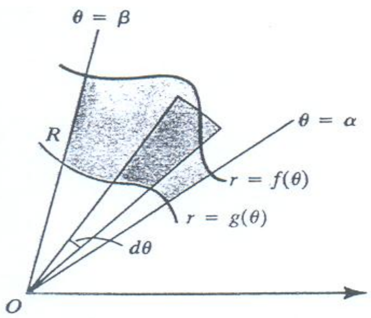

Fig. (2.24)

The area *A* of the region bounded by the graphs Fig. (2.24) of $r = f(\theta)$, $r = g(\theta)$ and the lines: $\theta = \alpha$ $\theta = \beta$ is,

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \left[f^2(\theta) - g^2(\theta) \right] d\theta$$

***Example* :30**

Find the area of the region bounded by the cardioid: $r = 2 + 2\cos\theta$

Solution

$$A = \int_{0}^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_{0}^{2\pi} \frac{1}{2} (2 + 2\cos\theta)^2 \, d\theta$$

Replace, $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$,

$$A = \int_{0}^{\pi} (6 + 8\cos\theta + 2\cos 2\theta) \, d\theta = \left(6\theta + 8\sin\theta + \sin 2\theta\right)_0^{\pi} = 6\pi$$

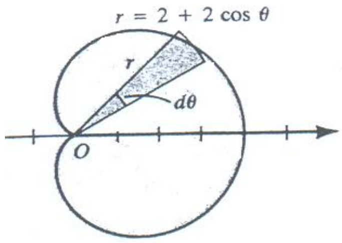

Fig.(2.25)

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***Example** :31*

Find the area of the region that is inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$.

Solution

The points of intersection are $(1, \pi/3), (1, -\pi/3)$.

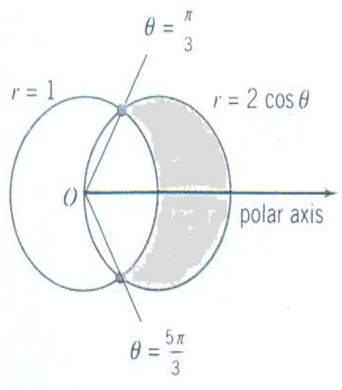

Fig. (2.26)

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(2\cos\theta)^2 - (1)^2\right] d\theta$$

$$= \int_{0}^{\pi/3}\left[(2\cos\theta)^2 - (1)^2\right] d\theta$$

$$= \int_{0}^{\pi/3}\left[4\cos^2\theta - 1\right] d\theta = \int_{0}^{\pi/3}\left[2(1+\cos 2\theta) - 1\right] d\theta$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} = 1.91$$

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***Example** :32*

Find the area of the region R that is inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 3$.

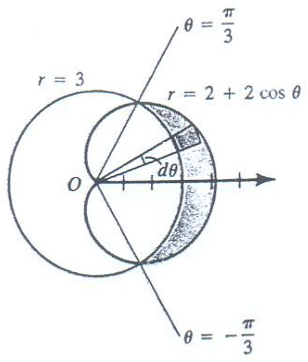

Fig. (2.27)

Solution

The points of intersection are $(3, \pi/3), (3, -\pi/3)$.

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(2 + 2\cos\theta)^2 - (3)^2\right] d\theta$$

$$= \int_{0}^{\pi/3}\left[4\cos^2\theta + 8\cos\theta - 5\right] d\theta$$

$$= \int_{0}^{\pi/3}\left[2(1+\cos 2\theta) + 8\cos\theta - 5\right] d\theta = \frac{9}{2}\sqrt{3} - \pi = 4.65$$

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***Example** :33*

Find the area of the region R that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$

Solution

The points of intersection are $(1, \pi/2), (1, -\pi/2)$.

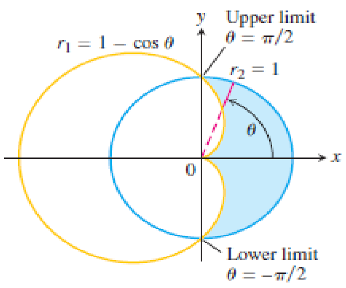

Fig. (2.28)

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1)^2 - (1 - \cos\theta)^2\right] d\theta$$

$$= (2)\left(\frac{1}{2}\right)\int_{0}^{\pi/2}\left[(1)^2 - (1 - \cos\theta)^2\right] d\theta = \int_{0}^{\pi/2}\left[2\cos\theta - \cos^2\theta\right] d\theta$$

$$= \int_{0}^{\pi/2}\left[2\cos\theta - \frac{1+\cos 2\theta}{2}\right] d\theta = \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi/2} = 2 - \frac{\pi}{4}$$

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Example :34

Find the area of the region bounded by the graph of the polar equation:

$$r^2 = 9\cos 2\theta$$

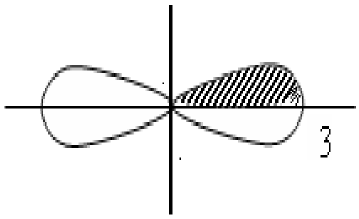

Fig. (2.29)

Solution

$$A = \frac{1}{2} \times 4 \int_0^{\pi/4} 9\cos 2\theta \, d\theta$$

$$= 18 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{18}{2}[1-0] = 9 .$$

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Example :35

Find the area of the region between the inner and outer, loops of the ***Limacons*** $r = 1 - 2\cos\theta$

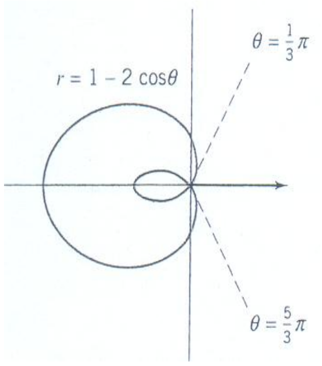

Fig. (2.30)

Solution

It is easy to verify that r = 0 when $\theta = \pi/3$ and when $\theta = 5\pi/3$. The outer loop is formed by having θ increase from $\pi/3$ to $5\pi/3$. Thus the area within outer loop :

$$A_1 = \int_{\pi/3}^{5\pi/3} \frac{1}{2}(1 - 2\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{5\pi/3} [1 - 4\cos\theta + 4\cos^2\theta] \, d\theta$$

$$= \frac{1}{2}\left(3\theta - 4\sin\theta + \sin 2\theta\right)_{\pi/3}^{5\pi/3} = 2\pi + \frac{3\sqrt{3}}{2}$$

The lower half of the inner loop is formed when θ increases from 0 to $\pi/3$, and the upper half when θ increases from $5\pi/3$ to 2π (verify this). Therefore, we have area within inner loop :

$$A_2 = \int_0^{\pi/3} \frac{1}{2}(1 - 2\cos\theta)^2 \, d\theta + \int_{5\pi/3}^{2\pi} \frac{1}{2}(1 - 2\cos\theta)^2 \, d\theta$$

$$= \frac{\pi}{2} - \frac{3\sqrt{3}}{4} + \frac{\pi}{2} - \frac{3\sqrt{3}}{4} = \pi - \frac{3\sqrt{3}}{2}$$

Thus, $A = A_1 - A_2$

$$= \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3} = 8.34$$

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Example :36

Find the area of the region bounded by the circle $r = 2\sin\theta$ and the limacons $r = 3/2 - \sin\theta$.

Solution

The points of intersection are $(1, \pi/6), (1, -5\pi/6)$.

From the symmetry of the region,

$$A = 2\left\{\frac{1}{2}\int_{-0}^{\pi/6}[2\sin\theta]^2\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2}\left[\frac{3}{2} - \sin\theta\right]^2 d\theta\right\}$$

$$= \frac{5\pi}{4} - \frac{15\sqrt{3}}{8} = 0.68$$

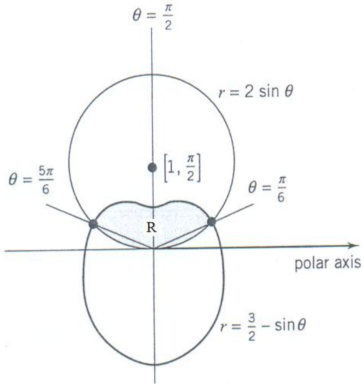

Fig. (2.31)

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<u>Exercise</u> (2 –4)

(I) Find the area of the region bounded by the graph of the polar equation,

(1) $r = 2\cos\theta$ (2) $r = 5\sin\theta$ (3) $r = 1 - \cos\theta$

(4) $r = 6 - 6\sin\theta$ (5) $r^2 = 9\cos 2\theta$ (6) $r^2 = 4\sin 2\theta$

(II) Find the area of the region R.

(1) $R = \left\{(r, \theta) : 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le e^{\theta}\right\}$

(2) $R = \left\{(r, \theta) : 0 \le \theta \le \pi,\ 0 \le r \le e^{2\theta}\right\}$

(3) $R = \left\{(r, \theta) : 0 \le \theta \le \pi,\ 0 \le r \le 2\theta\right\}$

(III) Find the area of the region bounded by one loop of the graph of the polar Equation,

(1) $r^2 = 4\cos 2\theta$ (2) $r = 2\cos 3\theta$ (3) $r = \sin 6\theta$

(4) $r = 3\cos 5\theta$ (5) $x = 0,\ y = 0$, $x = 4$ and $x^2 + y^2 = 25$

(IV) Find the area of the region that is inside the graphs of both equations,

(1) $r = 2 + 2\sin\theta,$ $r = 1$ (2) $r = 2\sin\theta$ $r = 1$

(3) $r^2 = 4\cos 2\theta$, $r = 1$ (4) $r = 2\sin\theta$, $r = 2\cos\theta$

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II-4 Polar Equations Of Conic Sections

The conic sections in Cartesian coordinates were studied in the first chapter, Polar coordinates are especially important in astronomy and astronautical engineering because satellites, moons, planets, and comets all move approximately along ellipses, parabolas, and hyperbolas that can be described with a single relatively simple polar coordinate equation.

We develop that equation here after first introducing the idea of a conic section's eccentricity. The eccentricity reveals the conic section's type (circle, ellipse, parabola, or hyperbola) and the degree to which it is "squashed" or flattened.

We can define the conic sections in a more general form as " The set of all points moves such that its distance from a fixed point to its distance from a fixed line is a constant ratio". The fixed point is called **focus "F"**, the fixed line is called **directrix "d"**, and the constant ratio is called **eccentricity "e".**

***Definition* : 3**

$$\text{Eccentricity} \quad e = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

(1) The eccentricity of the ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} < 1$$

(2) The eccentricity of the hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$$

(3) The eccentricity of the parabola :is $e = 1$

To find the polar form for the conic sections, Let $P(r, \theta)$ is the point moving according to the definition, F is the fixed point at origin and *L* is the fixed line.

From Fig. (2.32), $d(P, F) = r$, $d(P, Q) = d - r\cos\theta$,

Where d is the distance between *F* and *L*.

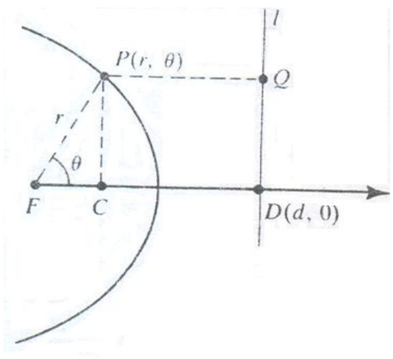

Fig (2.32)

Now the eccentricity : $e = \dfrac{d(P, F)}{d(P, Q)} = \dfrac{r}{d - r\cos\theta}$

$$r = de - re\cos\theta \quad \text{or} \quad r + re\cos\theta = de \text{ , then: } r = \frac{de}{(1 + e\cos\theta)}$$

***Theorem* : 8**

A polar equation, that has one of the forms,

$$r = \frac{de}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{de}{1 \pm e\sin\theta}$$

is a conic section with *major axis* along the polar axis or the line $\theta = \pi/2$ respectively. The conic section is a Parabola if $e = 1$, Ellipse if $0 < e < 1$, or Hyperbola if $e > 1$.

***Example* :37**

Describe and sketch the graph of the polar equation : $r = \dfrac{10}{3 + 2\cos\theta}$.

Solution

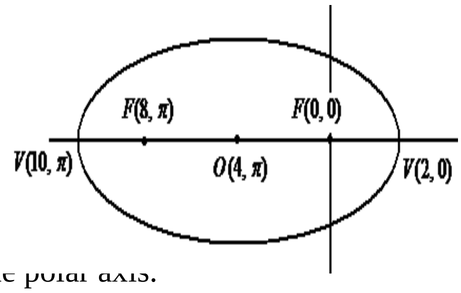

Fig. (2.33)

Divide both numerator and denominator by 3,

$$r = \frac{10/3}{1 + \frac{2}{3}\cos\theta} \text{ ; then } e = \frac{2}{3} < 1$$

Thus, it is an ellipse with major axis along the polar axis.

To find the vertices we put $\theta = 0$ and $\theta = \pi$.

$\theta = 0$, $r = 2$, then $V(2, 0)$,

$\theta = \pi$, $r = 10$, then $V(10, \pi)$,

then $2a = 12$ or $a = 6$. The center of the ellipse at $O(4, \pi)$

since $e = c/a$, $e = 2/3$, $a = 6$, then $c = 4$, and $b = \sqrt{a^2 - c^2} = \sqrt{20}$. The foci are $F(0, 0)$ and $F(8, \pi)$

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Example :38

Describe and sketch the graph of the polar equation : **:** $r = \dfrac{12}{6+2\sin\theta}$.

Solution

$$r = \frac{12}{6+2\,\mathrm{Sin}\,\theta} = \frac{2}{1+\frac{1}{3}\mathrm{Sin}\,\theta}, \text{ then : } e = \frac{1}{3} < 1$$

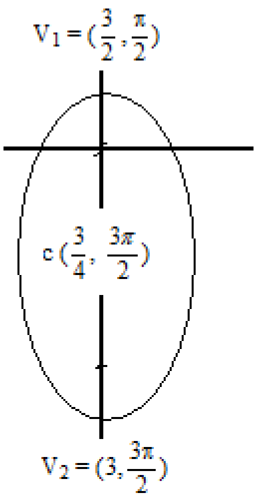

Fig. (2.34)

The conic section is Ellipse with major axis, the *y*-axis
To find the vertices we put $\theta = \pi/2$ and $\theta = 3\pi/2$.

$$\theta = \frac{\pi}{2} \Rightarrow r = \frac{12}{6+2} = \frac{12}{8} = \frac{3}{2}, \quad V_1 = (\frac{3}{2}, \frac{\pi}{2})$$

$$\theta = \frac{3\pi}{2} \Rightarrow r = \frac{12}{6-2} = 3, \quad V_2 = (3, \frac{3\pi}{2})$$

$$2a = \frac{3}{2} + 3 = \frac{9}{2}, \text{ then } \quad a = \frac{9}{4}$$

$$c = ea = \frac{1}{3} \times \frac{9}{4} = \frac{3}{4}$$

$$b^2 = a^2 - c^2 = \frac{81}{16} - \frac{9}{16} = \frac{9}{2}, \text{ then } \quad b = \frac{3}{\sqrt{2}}$$

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Example :39

Describe and sketch the graph of the polar equation : $r = \dfrac{10}{2 + 3\sin\theta}$

Solution

Divide both numerator and denominator by 2, $r = \dfrac{5}{1 + \frac{3}{2}\sin\theta}$; then $e = 3/2 > 1$.

Thus, it is hyperbola with major axis along the line $\theta = \pi/2$.

To find the vertices we put $\theta = \pi/2$ and $\theta = 3\pi/2$.

$\theta = \pi/2$, $r = 2$, then $V(2, \pi/2)$,

$\theta = 3\pi/2$, $r = -10$, then $V(-10, 3\pi/2)$,

then $2a = 8$ or $a = 4$. The center of the hyperbola at $O(-6, 3\pi/2)$

since $e = c/a$, $e = 3/2$, $c = 6$,

then , $b = \sqrt{c^2 - a^2} = \sqrt{20}$.

The foci are $F(0, \pi/2)$ and $F(-12, 3\pi/2)$.

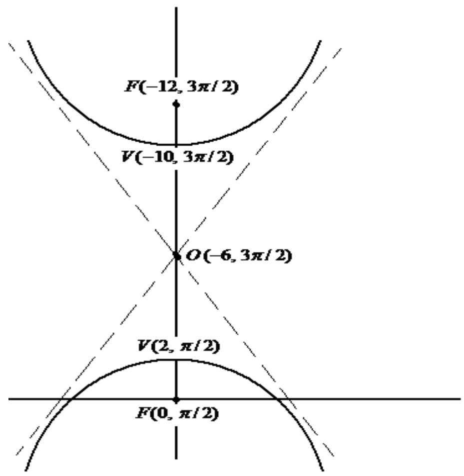

Fig. (2.35)

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***Example* :40**

Describe and sketch the graph of the polar equation:

$$r = \frac{15}{4 - 4\cos\theta}$$

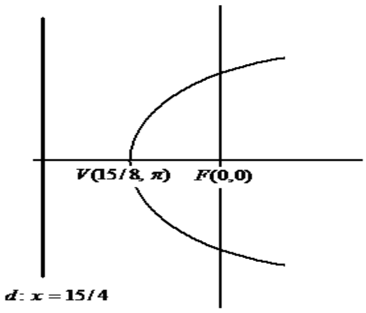

Fig.(2.36

Solution

Divide both numerator and denominator by 4,

$$r = \frac{15/4}{1 - \cos\theta} \;; \quad \text{then} \quad e = 1.$$

The graph is a parabola with major axis along the polar axis.

$\theta = 0, \quad r$ undefined

$\theta = \pi, \; r = 15/8$, then $V(15/8, \pi)$.

The parabola open right with vertex at $V(15/8, \pi)$, and $d = 2p = 15/4$, *i.e.* $p = 15/8$. Then the focus $F(0,0)$ at the pole (origin), and the directrix $d: x = 15/4$

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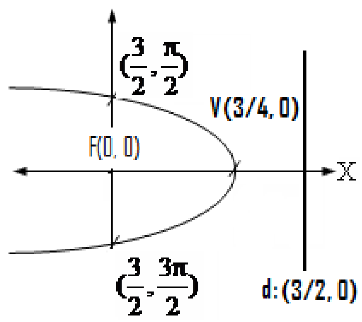
***Example* :41**

Describe and sketch the graph of the polar equation: $r = \dfrac{3}{2 + 2\cos\theta}$

Solution

$$r = \frac{3}{2 + 2\cos\theta} = \frac{3/2}{1 + \cos\theta}, \quad \text{then} \quad e = 1 .$$

The conic section is Parabola with vertex at the *polar-axis*.



$\theta = 0 \;\Rightarrow\; r = \dfrac{3}{4}$, $V = (\dfrac{3}{4}, 0)$

$\theta = \dfrac{\pi}{2} \;\Rightarrow\; r = \dfrac{3}{2}$

$\theta = \dfrac{3\pi}{2} \;\Rightarrow\; r = \dfrac{3}{2}$

de = 3/2, then d = 3/2, i.e.

the distance between the directrix and focus = $2p$ = 3/2.

Then the focus $F(0,0)$ at the pole (origin), and the directrix $d: x = 3/2$

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Example :42

Find an equation in x and y that has the same graph as the polar equation , $r = \dfrac{15}{4 - 4\sin\theta}$

Solution

$$r(4 - 4\sin\theta) = 15 \quad \text{or} \quad 4r = 4r\sin\theta + 15$$

$$4\sqrt{x^2 + y^2} = 4y + 15$$

Squaring both sides and simplifying,

$$16(x^2 + y^2) = 16y^2 + 120y + 225 \quad \text{or} \quad 16x^2 = 120y + 225$$

Which is an equation of parabola.

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Example :43

Find a polar equation of the conic with a focus at the pole, eccentricity $e = 1/2$ and directrix $r = -3\sec\theta$.

Solution

The directrix : $r = -3\sec\theta$, $r\cos\theta = -3$ *i.e.* $x = -3$.

Since the focus at the pole, then the distance d between the focus and directrix $d = 3$. Then,

$$r = \frac{de}{1 - e\cos\theta} = \frac{(3)(1/2)}{1 - (1/2)\cos\theta} = \frac{3}{2 - \cos\theta}$$

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E x e r c i s e (2 – 5)

(I) Describe and sketch the graph of the polar equations,

(1) $r = \dfrac{12}{6+2\sin\theta}$. (2) $r = \dfrac{12}{2+6\cos\theta}$. (3) $r = \dfrac{3}{2+2\cos\theta}$.

(4) $r = \dfrac{2}{3+3\sin\theta}$. (5) $r = \dfrac{4}{\cos\theta-2}$. (6) $r = \dfrac{2}{\cos\theta-4}$.

(7) $r = \dfrac{4\sec\theta}{2\sec\theta-1}$ (8) $r = \dfrac{2\sec\theta}{4\sec\theta+1}$ (9) $r = \dfrac{6}{4-\cos\theta}$.

(II) Find the equations in *x* and *y* for the following polar equations

(1) $r = \dfrac{12}{6+2\sin\theta}$. (2) $r = \dfrac{12}{2+6\cos\theta}$. (3) $r = \dfrac{3}{2+2\cos\theta}$.

(4) $r = \dfrac{2}{3+3\sin\theta}$. (5) $r = \dfrac{6}{4-\cos\theta}$. (6) $r = \dfrac{6}{1+4\cos\theta}$.

(III) Find a polar equations of the conic with focus at the origin and the given eccentricity and equation of the directrix.

(1) $e = \dfrac{1}{3}$, $r = 2\sec\theta$ (2) $e = \dfrac{1}{2}$, $r = 3\csc\theta$

(3) $e = 1$, $r\cos\theta = 5$ (4) $e = 1$, $r\sin\theta = 4$

(5) $e = 3$, $r = -4\sec\theta$ (6) $e = 2$, $r\sin\theta = -3$

(7) $e = \dfrac{2}{5}$, $r = 4\csc\theta$ (8) $e = \dfrac{2}{3}$, $r\cos\theta = 3$

(IV) Find the slope of the tangent line to the conic at the point that corresponding to the given θ.

(1) $r = \dfrac{12}{6+2\sin\theta}$, $\theta = \dfrac{\pi}{6}$ (2) $r = \dfrac{12}{2-6\cos\theta}$, $\theta = \dfrac{\pi}{3}$

(3) $r = \dfrac{12}{2-6\cos\theta}$, $\theta = \dfrac{\pi}{2}$ (4) $r = \dfrac{12}{6-2\sin\theta}$, $\theta = \dfrac{\pi}{2}$

(5) $r = \dfrac{12}{2+6\cos\theta}$, $\theta = \dfrac{\pi}{3}$ (6) $r = \dfrac{12}{6-2\sin\theta}$, $\theta = 0$

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CHAPTER 3

MULTIVARIABLE FUNCTIONS
(FUNCTIONS OF SEVERAL VARIABLES)

Functions with two or more independent variables appear more often in science than functions of a single variable, and their calculus is even richer. Their derivatives are more varied and more interesting because of the different ways in which the variables can interact. Their integrals lead to a greater variety of applications. The mathematics of these functions is one of the finest achievements in science.

1- Functions Of Several Variables

In this section, we define functions of more than one independent variable and discuss ways to graph them , we start by definitions to the function of several variables.

Definition **: 1**

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real valued function f on D is a rule that assigns a real number, $w = f(x_1, x_2, \ldots, x_n)$ to each element in D. The set D is the function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable of f and f is said to be a function of the n independent variables $(x_1, x_2, \ldots, x_n)$.

If f is a function of two independent variables, we usually call the independent variables *x* and *y* and the domain of f is a region in the *xy*-plane. If f is a function of three independent variables, we call the variables *x*, y, and z and the domain is a region in space. If the domain is not specified, then it is automatically taken to be the largest set for which the expression defining f is meaningful.

<u>Example</u> : 1

Let $f(x,y) = \sqrt{y - x^2}$. Find the domain of f and sketcl

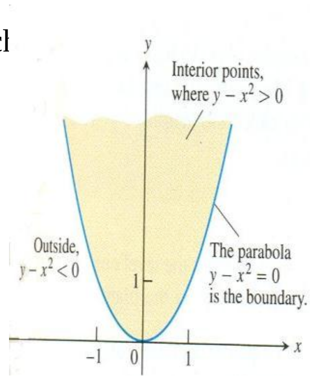

Fig. (3.1)

<u>Solution</u>

The domain D is the set of all pairs (*x*, *y*) with:

$y - x^2 \geq 0$ or $y \geq x^2$.

The parabola: $y = x^2$, is the boundary of the domain.

The points above the parabola make up the domain's interior.

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<u>Example</u> : 2

Find the domain of the function $f(x,y) = 100 - x^2 - y^2$

<u>Solution</u>

The function is defined for all values of *x* and *y*,

i.e. The domain is the entire $xy - \text{plane}$.

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<u>Example</u> : 3

Find the domain of the function $f(x,y,z) = \dfrac{\cos^{-1} z}{\left(1 + \sqrt{x^2 + y^2 - 1)}\right)}$

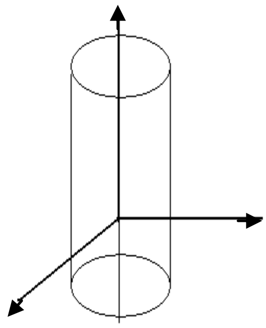

Fig. (3.2)

<u>Solution</u>

The domain consists of all triples (x, y, z) with $x^2 + y^2 \geq 1$ and $|z| \leq 1$. This is the region outside of cylinder and bounded by the two planes $z = -1$, $z = 1$ as in Fig. (3.2).

A function of three variables is defined just as in the above definition, except that the domain is the set of ordered triples (x, y, z)) and the values of f are denoted by $f(x, y, z)$.

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2- Graphs Of Functions Of Two Variables

Definition **: 2**

> The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x,y))$ in space for (x, y) in the domain of f, is called the graph of *f*. The **graph** of f is also called the **surface** $z = f(x, y)$.

The graph of an equation in the three variables *x, y,* and *z* is a surface in space.

The graph of a linear equation: $ax + by + cz = d$ is a plane. The simplest planes are the planes : $x = x_0$, $y = y_0$ and $z = z_0$. They are represented in Fig. (3.3)

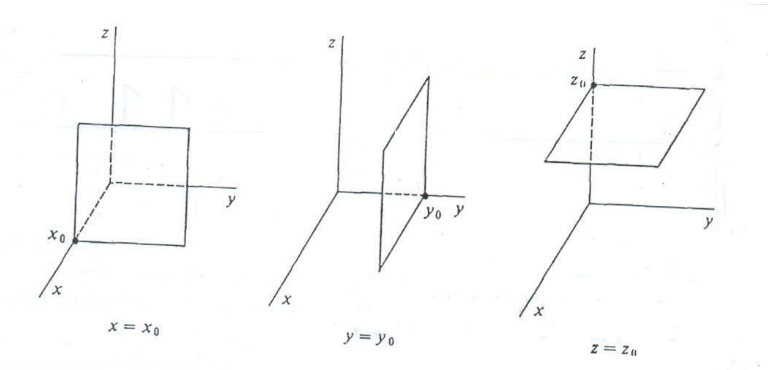

Fig. (3.3)

The graph of a second degree equation in *x, y,* and *z* is a surface in space called a quadratic surface. These surfaces correspond to the conic sections in the plane. There are several types of quadratic surfaces. We shall present each of them in its simplest form.

Example **: 4**

The **ellipsoid** : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

cuts the coordinate axis at

$(\pm a, 0, 0)$, $(0, \pm b, 0)$ and $(0, 0, \pm c)$. Fig. (3.4)

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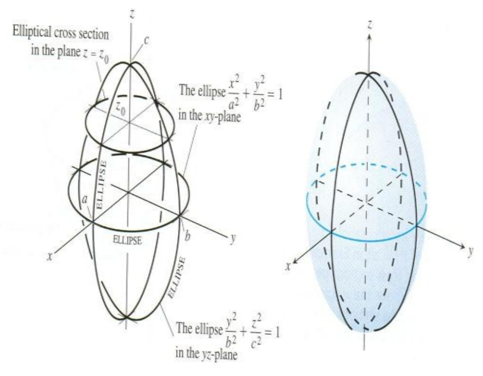

Fig. (3.4)

If any two of the semi-axes *a, b,* and *c* are equal, the surface is an **ellipsoid of revolution.** If all three are equal, the surface is a **sphere**.

***<u>Example</u>* : 5**

The **elliptic paraboloid :** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$,

Is symmetric with respect to the planes $x = 0$ and $y = 0$. The only intercept on the axes is the origin. Except for this point, the surface lies above or entirely below the *xy* – plane, depending on the sign of *c*. Fig. (3.5)

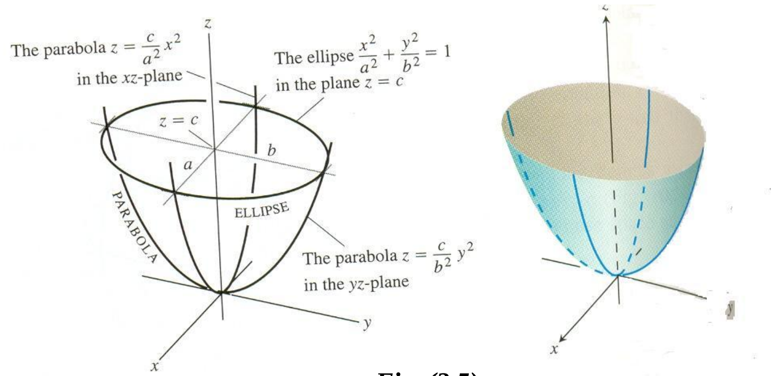

Fig. (3.5)

If $a = b$, the elliptic paraboloid is called a **circular paraboloid.** In this case the cross sections of the surface by planes perpendicular to the *z*-axis are circles centered on the *z*-axis.

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***<u>Example</u>* : 6**

The **elliptic cone:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

Is symmetric with respect to the three coordinate planes. Fig. (3.6)

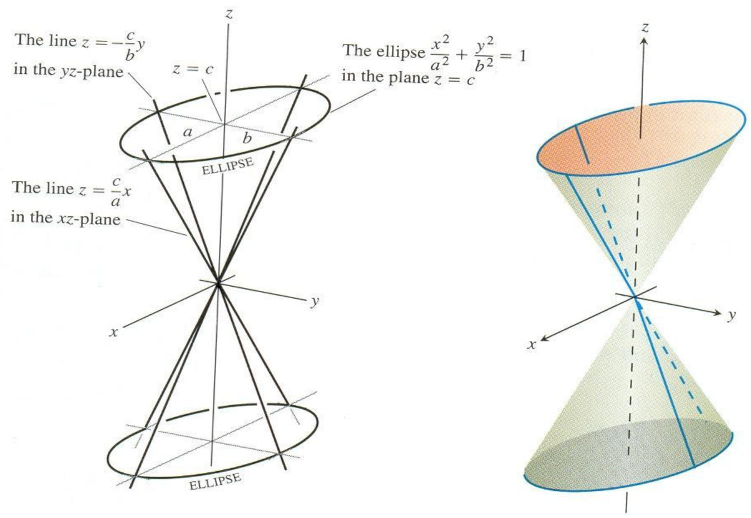

Fig. (3.6)

If $a = b$, the cone is a right circular cone.

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Example : 7

The **hyperboloid of one sheet :** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$,

is symmetric with respect to each of the three coordinate planes . The plane $z = z_0$ cuts the surface in an **ellipse** with center on the *z*-axis and vertices on one of the hyperbola. Fig. (3.7)

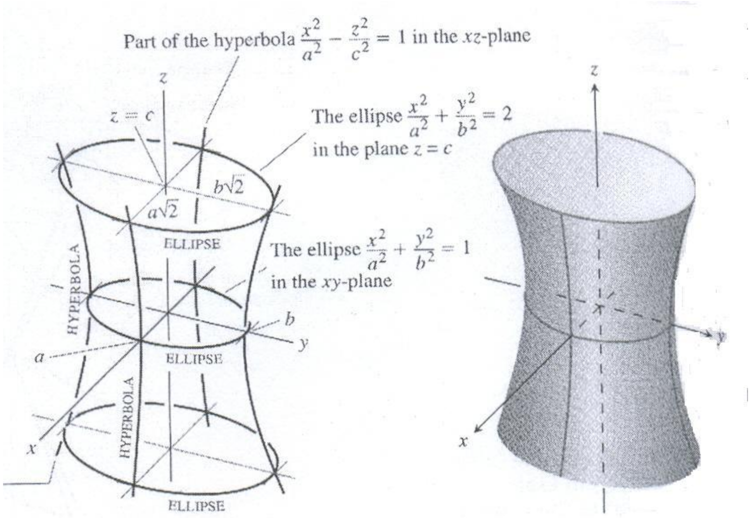

Fig. (3.7)

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Example : 8

The **hyperboloid of two sheets:** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$,

is symmetric with respect to each of the three coordinate planes. The plane $z = 0$ does not intersect the surface, in fact, for a horizontal plane to intersect the surface, we must have $|z| \geq c$.

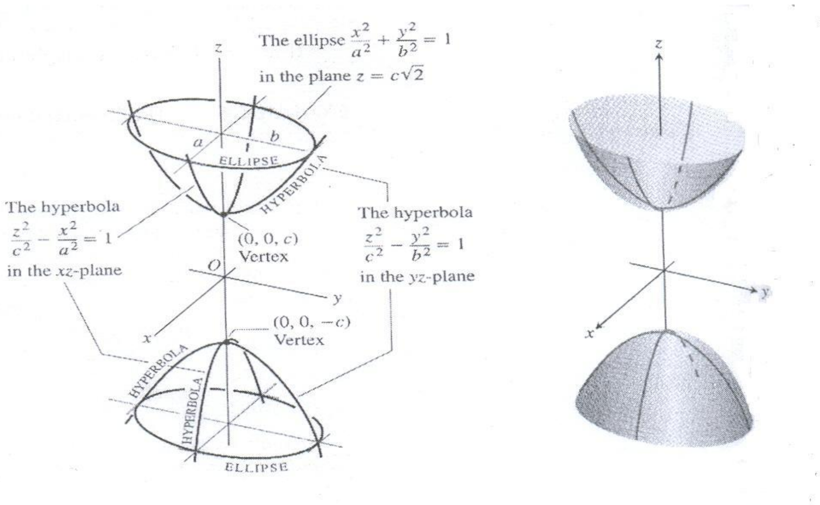

Fig. (3.8)

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Example **: 9**

The **hyperbolic paraboloid:** $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c}$; $c > 0$,

has symmetry with respect to the planes $x = 0$ and $y = 0$.

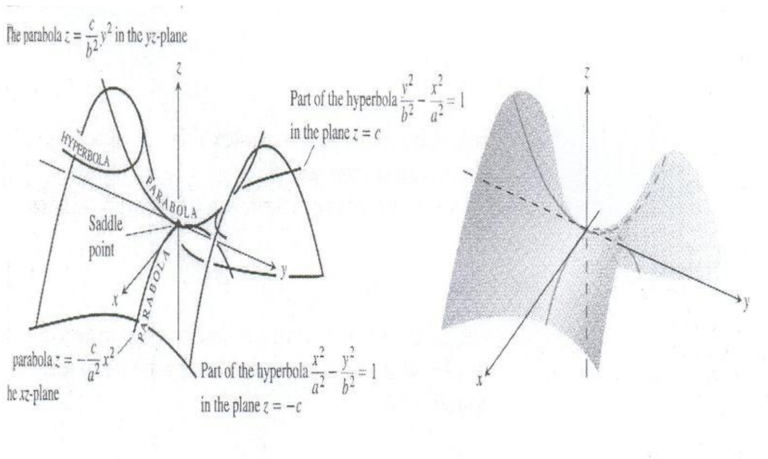

Fig. (3.9)

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E x e r c i s e (3– 1)

Determine the domain of the function $f(x, y)$ and sketch it.

(1) $f(x, y) = \sqrt{1 - x^2 - y^2}$

(2) $f(x, y) = 3\sqrt{x} + \sqrt{y}$

(3) $f(x, y) = \sqrt{y^2 - 4x^2 - 16}$

(4) $f(x, y) = \sqrt{x - y + 2}$

(5) $f(x, y) = \sqrt{x^2 - 4y^2 - 25y}$

(6) $f(x, y) = \dfrac{1}{x^2 + y^2 - 4}$

(7) $f(x, y) = \ln(x + y)$

(8) $f(x, y) = \sqrt{x - 1} + \sqrt{y}$

(9) $f(x, y) = \sin^{-1}(x + y)$

(10) $f(x, y) = \ln(x^2 + y^2 - 1)$

(11) $f(x, y) = \dfrac{1}{\sqrt{1 - xy}}$

(12) $f(x, y) = \cos^{-1}(x - y)$

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3- Limits And Continuity

This section treats limits and continuity for multivariable functions. If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0).

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0) and write, $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$

***Theorem* : 1**

If $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x, y) = M$, Then,

1. Sum & Difference Rules: $\lim_{(x,y)\to(x_0,y_0)} (f \pm g) = L \pm M$.
2. Product Rule: $\lim_{(x,y)\to(x_0,y_0)} (f \bullet g) = L \bullet M$.
3. Constant Multiple Rule : $\lim_{(x,y)\to(x_0,y_0)} K f = K L$, K is a constant
4. Quotient Rule : $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$, $M \neq 0$.
5. Power Rule : If m and n are integers, then:

$$\lim_{(x,y)\to(x_0,y_0)} [f(x, y)]^{m/n} = L^{m/n} \quad \text{provided } L^{m/n} \text{ is a real number}$$

.

***Example* : 10**

Evaluate the following limits,

(i) $\lim_{(x,y)\to(2, 3)} (xy - 2) = 6 - 2 = 4$.

(ii) $\lim_{(x,y)\to(1, 4)} (x^2 y + 3x y^2) = 4 + 48 = 52$.

(iii) $\lim_{(x,y)\to(\pi/2, 1)} (4x^2 + y^3 \sin x) = \pi^2 + 1$.

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If the limit at origin $(x, y) \to (0, 0)$, and the value of limit $= 0/0$, we can use a simple method by considering the limits through the set of all lines passing through the origin. If this limit depend on the slope of the lines, then the limit depend on the path and so the limit does not exist.

Example : 11

Show that the following limits does not exist

(i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$, (ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

Solution

(i) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x(mx)}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}.$$

The limit depend on the slope of the lines *m*, *i.e.* the limit depend on the path and so the limit does not exist.

(ii) $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}$$

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The limit depend on the slope of the lines *m*, *i.e.* the limit depend on the path and so the limit does not exist.

As with functions of a single variable, continuity is defined in terms of limits.

Definition : 3

A function $f(x, y)$ is continuous at the point (x_0, y_0), if :

1. f is defined at (x_0, y_0) ,
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exist
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

A function is continuous on a region *D*, if it is continuous at each point in D.

One of the consequences of Theorem (3.3.1) is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, products, constant multiples, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

If $z = f(x, y)$ is a continuous function of *x* and *y*, and $w = g(z)$ is a continuous function of z, then the composite $w = g(f(x, y))$ is continuous. Thus, the polynomials, exponential, sine, cosine, and logarithmic functions are continuous at every point (x, y) .

As with functions of a single variable, the general rule is that composites of continuous functions are continuous. The only requirement is that each function be continuous where it is applied.

Example **: 12**

Find all points where the given function is continuous:

(i) $f(x) = \dfrac{x}{x^2 - y}$, (ii) $f(x) = \dfrac{x+3}{x^2 + y^2}$

Solution

(i) The function is a quotient of two polynomials (continuous functions), and so it is continuous at any point except at the denumerator equal zero. So the function is continuous at all point except at $y = x^2$.

(ii) The function is a quotient of two continuous functions, and so it is continuous at any point except at the denumerator equal zero. The denumerator equal zero at $(x, y) = (0, 0)$, then $f(x, y)$ is continuous for all values of (x, y) except at $(x, y) = (0, 0)$.

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Example **: 13**

Discuss the continuity of the following function

$$f(x) \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution

The function is also continuous at any point $(x, y) \neq (0, 0)$. At $(x, y) = (0, 0)$ the function is defined, $f(0, 0) = 0$, but the limit as $(x, y) \to (0, 0)$ does not exist and so the function is discontinuous at the origin.

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<u>Exercise</u> (3– 2)

(I) Find the limits of the following functions if it exists.

(1) $\lim\limits_{(x,y)\to(4,-2)} x^3\sqrt{y^3+2x}$ (2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2-2y^2+5}{x^2+y^2+3}$

(3) $\lim\limits_{(x,y)\to(0,2)} \dfrac{\sin xy}{x}$ (4) $\lim\limits_{(x,y)\to(1,0)} \dfrac{x\sin y}{x^2+1}$

(5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x^2-y^2}$ (6) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2-y}$

(7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ (8) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$

(9) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ (10) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+y^4}$

(11) $\lim\limits_{(x,y)\to(0,0)} \cos\left(\dfrac{x^2+y^2}{x^2+y^2+1}\right)$ (12) $\lim\limits_{(x,y)\to(0,0)} \sin\left(\dfrac{xy}{x-y+3}\right)$.

(II) Discuss the continuity of the following functions,

(1) $f(x,y) = \dfrac{x^2-y^2+1}{x^2+y^2+3}$ (2) $f(x,y) = \dfrac{2x-3y+1}{x+y-1}$

(3) $f(x,y) = \dfrac{x^2-y^2+1}{x^2+y^2-3}$ (4) $f(x,y) = \dfrac{y\cos x}{1-e^y}$

(5) $f(x,y) = \dfrac{e^{x+y}+1}{1-\cos x}$ (6) $f(x,y) = \dfrac{x^2+y^2}{2-\cos xy}$

(7) $f(x,y) = \dfrac{x\sin y}{2x+y}$ (8) $f(x,y) = \dfrac{\cos(x+y)}{x-y^2}$

(9) $f(x,y)\begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

(10) $f(x,y)\begin{cases} \dfrac{x^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

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4- Partial Derivatives

4.1 First Order Partial Derivatives

As in functions of single variable $f(x)$, the derivative $f'(x)$ is defined as,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

An analogous procedure can be applied to functions of several variables, we can obtain ***partial derivatives*** for the function of several variables with respect to each independent variable.

Definition **: 4**

Let $f(x,y)$ be a function of two variables, the first partial derivatives of $f(x,y)$ with respect to *x* and *y* are the function f_x and f_y defined by :

$$f_x = \frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

$$f_y = \frac{\partial}{\partial y} f(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

In the definition of f_x , y is held fixed, only *x* is allowed to vary. If x is fixed and the only y is allowed to vary, then f_y is the derivative with respect to *y*.

We calculate $\frac{\partial f}{\partial x}$ by differentiating f with respect to *x* in the usual way while treating *y* as a constant, and we can calculate $\frac{\partial f}{\partial y}$ by differentiating *f* with respect to *y* in the usual way while holding *x* constant.

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable.

Example : 14

Find the *first partial derivatives* of the following functions

(i) $f(x, y) = x^3 y^2 - 2x^2 y + 3x + 2y + 1.$

(ii) $z = x y^2 e^{xy}$, (iii) $w = x y^2 z^3$

Solution

(i) $f_x(x, y) = 3x^2 y^2 - 4x y + 3$

$f_y(x, y) = 2x^3 y - 2x^2 + 2.$

(ii) $\frac{\partial z}{\partial x} = x y^2 \, y e^{xy} + y^2 e^{xy} = (x y^3 + y^2) e^{xy}$

$\frac{\partial z}{\partial y} = x y^2 \, x e^{xy} + 2 x y e^{xy} = (x^2 y^2 + 2 x y) e^{xy}$

(iii) $\frac{\partial w}{\partial x} = y^2 z^3, \quad \frac{\partial w}{\partial z} = 2 x y z^3, \quad \frac{\partial w}{\partial z} = 3 x y^2 z^2$

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Example : 15

Find the *first partial derivatives* of the following functions

(i) $f(x, y) = y \sin x y.$, (ii) $f(x, y) = \frac{x y}{y + \cos x}$

Solution

(i) $\frac{\partial f}{\partial x} = y (\cos x y) (y) = y^2 \cos x y$

$\frac{\partial f}{\partial y} = y (\cos x y) (x) + \sin x y = x y \cos x y + \sin x y$

(ii) $\frac{\partial f}{\partial x} = \frac{(y + \cos x)(y) - xy(-\sin x)}{(y + \cos x)^2} = \frac{y^2 + y\cos x + xy \sin x}{(y + \cos x)^2}$.

$\frac{\partial f}{\partial y} = \frac{(y + \cos x)(x) - x y(1)}{(y + \cos x)^2} = \frac{x \cos x}{(y + \cos x)^2}$

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Example : 16

Find $\frac{\partial z}{\partial x}$ for the function *z* defined in terms of *x* and *y* by the equation,

$y z - \ln z = x + y$

Solution

$\frac{\partial}{\partial x}(y z - \ln z) = \frac{\partial}{\partial x}(x + y)$

$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1$, *i.e.* $\left(y - \frac{1}{z}\right) \frac{\partial z}{\partial x} = 1$.,Then $\frac{\partial z}{\partial x} = \frac{z}{y z - 1}$

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4-2 Higher Order Partial Derivatives.

Higher order partial derivatives may be defined in a similar way, provided that the earlier ones are functions of (x, y) and are continuous at the point under consideration. This second partial derivatives of $f(x, y)$ are defined as follows.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx} \quad , \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx} \quad , \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$

The second partials derivatives f_{xy} & f_{yx} are called *mixed partials derivatives* and the following theorem illustrate their relations

***Theorem* : 2** **Euler's Theorem** **(** *The Mixed Derivative Theorem* **)**

If $f(x, y)$ and its partial derivatives f_x , f_y , f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example **: 17**

Find all second partial derivatives of the function: $f(x, y) = x^3 y + x^2 y^3 + x y$

Solution

$$\frac{\partial f}{\partial x} = f_x = 3x^2 y + 2xy^3 + y, \qquad \frac{\partial f}{\partial y} = f_y = x^3 + 3x^2 y^2 + x$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6xy + 2y^3$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 3x^2 + 6xy^2 + 1$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 3x^2 + 6xy^2 + 1$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6x^2 y$$

Note that in the above example, the partials derivatives f_{xy} & f_{yx} are equal.

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In the same way the third and higher partial derivatives can defined, for examples

$$\frac{\partial^3 f}{\partial x\, \partial^2 y} = f_{yyx} \quad \text{and} \quad \frac{\partial^4 f}{\partial^2 x\, \partial^2 y} = f_{yyxx}$$

***Example* : 18**

If $w = \cos(x - y) + \ln(x + y)$, show that: $\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0$

Solution

$$\frac{\partial w}{\partial x} = -\sin(x - y) + \frac{1}{x + y} \quad , \quad \frac{\partial w}{\partial y} = \sin(x - y) + \frac{1}{x + y}$$

$$\frac{\partial^2 w}{\partial x^2} = -\cos(x - y) - \frac{1}{(x + y)^2} \quad , \quad \frac{\partial^2 w}{\partial y^2} = -\cos(x - y) - \frac{1}{(x + y)^2}$$

$$\text{L.H.S.} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$$

$$= \left(-\cos(x - y) - \frac{1}{(x + y)^2}\right) - \left(-\cos(x - y) - \frac{1}{(x + y)^2}\right) = 0 = \text{R.H.S.}$$

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Exercise (3– 3)

(I) Find all the first partial derivatives of the given functions,

(1) $f(x,y) = 3x^3y + 2xy^2$ (2) $f(x,y) = \sqrt{x} + \sqrt{y} - 10$

(3) $f(x, y) = \sqrt{x^2 + y^2}$ (4) $f(x,y) = \cos xy + \sin x^2 + e^{2y}$

(5) $f(x,y,z) = x\cos(y/z)$ (6) $f(x,y) = x\sin y - y\tan x + xy$

(7) $f(x,y,z) = \sin^{-1}(xyz)$ (8) $f(x,y) = \sqrt[3]{x^2} - 2x^2y^3 + \sqrt[5]{xy^2}$

(9) $g(r, s, t) = r s \sec t$ (10) $f(x,y,z) = x\cos yz - y\sin xz$

(11) $f(x, y) = \dfrac{e^{xy}}{y\sin x}$ (12) $f(x,y,z) = xy\,e^{yz} - \ln xyz$

(II) Find the required partial derivatives at the indicated points

(1) $w = x^2 + y^2 - 2xy\cos z$. $\dfrac{\partial^2 w}{\partial x\partial z}(0, 1, \pi/6)$

(2) $w = xz\,e^y - yze^x + xy\,e^z$. $\dfrac{\partial^2 w}{\partial x\partial z}(0, 1, \pi/6)$

(3) $u(x, y, z) = \dfrac{x^2 + y^2}{x\,z}$, $\dfrac{\partial^2 u}{\partial y\partial x}$ (1, 3, 1)

(4) $u(x, y, z) = \dfrac{x^3 + xy^2}{y\,z^3}$, $\dfrac{\partial^2 u}{\partial y\partial x}$ (2, 2, 1)

(5) $f(r, s, t) = e^{rs}\sin t$, $\dfrac{\partial^3 f}{\partial r\,\partial t\,\partial s}(3, 1, 0)$

(III) For the following functions, confirm that the mixed second order partial derivatives are equals,

(1) $f(x,y) = \ln(2x + 3y)$ (2) $f(x,y) = x\ln y + e^x + y\ln x$

(3) $f(x,y) = xy^2 + x^2y + x^3y^4$ (4) $f(x,y) = x\ln y + e^x + y\ln x$

(5) $f(x,y) = x\cosh y + x^2\sinh y$ (6) $f(x,y) = y^2\ln x + x^2e^y$

(IV) Show that $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(1) $u = x^2 - y^2$; $v = 2xy$

(2) $u = e^x \cos y$; $v = e^x \sin y$

(3) $u = \ln(x^2 + y^2)$; $v = 2\tan^{-1} y/x$

(V) Show that the following functions satisfy the *two-dimensional Laplace* equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

(1) $f(x, y) = e^{-2y} \cos 2x$, (2) $f(x, y) = \ln\sqrt{x^2 + y^2}$.

(VI) Show that the following functions satisfy the *third-dimensional Laplace* equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

(1) $f(x, y, z) = x^2 + y^2 - 2z^2$, (2) $f(x, y, z) = e^{3x+4y} \cos 5z$

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5- The Increments And Differentials

If $w = f(x, y)$ is a function of two variables *x* and *y*, then the symbols Δx and Δy denotes increments of *x* and *y* respectively. In terms of increments, the partial derivatives may be written as,

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

The increment Δw represents the change in the function value if (x, y) changes to $(x + \Delta x,\ y + \Delta y)$, then the increment Δw of $w = f(x, y)$ is, $\Delta w = f(x + \Delta x, y + \Delta y) - f(x, y)$

Definition **: 5**

Let $w = f(x,y)$, and let Δx and Δy denotes increments of x and y, respectively ,

(1) the **differential**s *dx* and *dy* of the independent variables x and y are

$dx = \Delta x$ and $dy = \Delta y$

(2) the **differential** dw of the dependent variable *w* is,

$$dw = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$$

If $w = f(x,y,z)$, then, the **differential** dw of the dependent variable w is,

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

where dx, dy and dz are the differentials of the independent variables x, y and z.
The extension to functions of more variables is similar.

Example **: 19**

If $w = 3x^2 - xy$, find *dw* and use it to approximate the change in *w* if (x, y) change from (1, 2) to (3.01, 1.98). How does this compare with the exact change in *w*?

Solution

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy = (6x - y)\, dx + (-x)\, dy$$

Substituting, $x = 1$, $y = 2$, $\Delta x = 0.01$ and $\Delta y = -0.02$, we obtain

$$dw = (6 - 2)(0.01) + (-1)(-0.02) = 0.06$$

To obtain the exact value,

$$dw = w(1.01, 1.98) - w(1, 2)$$

$$= 3(1.01)^2 - (1.01)(1.98) - 3(1)^2 + (1)(2) = 0.0605$$

Error $= 0.0605 - 0.06 = 0.0005$

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Example **: 20**

The radius and height of a right cylinder cone can be measured to be 12 and 36 cm respectively. If the measurement is accurate to within ± 0.05. Approximate the maximum possible error in the calculated volume of the cylinder.

Solution

The volume of the cone: $V = \frac{1}{3}\pi r^2 h$. The differential of *V* is:

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 \, dh.$$

The possible error in the radius measurement is: $dr = \pm(12)(0.0005) = \pm 0.06$

The possible error in the height measurement is: $dh = \pm(36)(0.0005) = \pm 0.18$

Therefore, the maximum error in computing the volume is approximately,

$$dV = \frac{2}{3}\pi(12)(36)(\pm 0.06) + \frac{1}{3}\pi(12)^2(\pm 0.18) = \pm 81.4 \text{ cm}^3.$$

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Example **: 21**

Calculate the approximate value of $(1.02)^{3.01}$.

Solution.

Consider the function: $z = x^y$,with, $x = 1$ and $y = 3$, and

$dx = 0.02$ and $dy = 0.01$, then,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = y\, x^{y-1} dx + x^y \ln x \, dy$$

$$= (3)(1)^2(0.02) + (1)^3 \ln(1)(0.01) = 0.06$$

Hence, $z = (1.02)^{3.01} = z_0 + dz = (1)^3 + 0.06 = 1.06$

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Exercise (3– 4)

(I) Find the differentials for the given functions

(1) $z = x^2 - 3x\,y + y^2$ (2) $z = x^3 + x^2 y^2 - y^5$ (3) $z = \tan^{-1}(x^2 y)$

(4) $z = e^{y^2} \sin^{-1} x^3$ (5) $w = xy^2 \ln z$ (6) $w = x\,y \ln z + xz \ln y$

(7) $w = x \cos y + y \sin z$ (8) $w = z \sinh^{-1} x + x \operatorname{sech}^{-1} y$

(II) Approximate the change in the function f as the independent variables changes from P to Q.

(1) $f(x,y) = x^2 + 2xy - 4x$, P(1, 2) , Q(1.02, 2.04)

(2) $f(x,y) = x^{1/3}\, y^{1/2}$, P(8, 9) , Q(7.78, 9.03)

(3) $f(x,y) = \dfrac{x + y}{xy}$, P(1, 2) , Q(1.01, 2.02)

(4) $f(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$ P(1,4,9), Q(0.97,4.1,8.5)

(5) $f(x,y,z) = xy + yz + xz$ P(1,2,3), Q(0.8,2.05,2.96)

(III) Use differentials to approximate the given problems

(1) Estimate the value of $\sin\left(\dfrac{89\,\pi}{180}\right) \cos^2\left(\dfrac{29\pi}{180}\right)$.

(2) Estimate the value of $\tan^2\left(\dfrac{43\pi}{180}\right)$.

(3) Estimate the value of $\sqrt{101.2} / \sqrt[3]{26.3}$

(4) Estimate the value of $\sqrt{81.1}\ x\ \sqrt[3]{7.8}$

(5) The dimensions of a rectangular parallelepiped are measured as 6, 2 and 5 inches with possible error in measurement of 1/2%. Approximate the maximum error in computing the surface area.

(6) Calculate the approximate value of $\sqrt{(9.02)^2 /(4.03)^2}$

(7) Calculate the approximate value of $(3.02)^2\ x\ (0.97)^2$

(8) Calculate the approximate value of $(4.02)^2 / (3.97)^2$

(9) Calculate the approximate value of $\sqrt{(4.05)^2 + (2.93)^2}$

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6- Chain Rules and Implicit Differentiation

6.1 Chain Rules

In the first course of "Calculus" we considered the differentiable functions $y = f(x)$ and $x = g(t)$, then the chain rule is, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

The analogous for functions of two or three variables is given in the following theorems

***Theorem* : 3 (Rule 1)**

If $z = f(x, y)$ and $x = x(t),\ y = y(t)$ are all differentiable functions, then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

Similar statements hold for functions of three or more variables, in fact if, $w = f(x, y, z, \ldots, s)$ is a differentiable function of any number of variables, and each variable, is differentiable function of one variable *t* , then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \ . \ . \ . + \frac{\partial w}{\partial s}\frac{ds}{dt}.$$

***Example* : 22**

Find $\frac{df}{dt}$ for the following functions

(i) $f(x, y) = x^2 y + e^{2x-y}$, $x = \cos t$, $y = 4t^3$.

(ii) $f(x, y) = \tan^{-1} x y$, $x = \tan t$, $y = 1/t$.

(iii) $f(x, y, z) = \sin(xyz)$, $x = 1/t$, $y = \ln t$, $z = t$.

Solution

(i) $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 2e^{2x-y})(-\sin t) + (x^2 - e^{2x-y})(12t^2)$$

$$= -8t^3 \cos t \sin t - 2\sin t\, e^{2\cos t - 4t^3} + 12t^2 \cos^2 t - 12t^2 e^{2\cos t - 4t^3}$$

(ii) $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{y}{1+(xy)^2}\sec^2 t + \frac{x}{1+(xy)^2}\left(\frac{-1}{t^2}\right)$$

$$= \frac{t \sec^2 t - \tan t}{t^2 + \tan^2 t}.$$

(iii) $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= yz\cos(xyz)\left(\frac{-1}{t^2}\right) + xz\cos(xyz)\left(\frac{1}{t}\right) + xy\cos(xyz)$$

$$= \cos(xyz)\left[\frac{-yz}{t^2} + \frac{xz}{t} + xy\right] = \frac{1}{t}\cos(\ln t)$$

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***Theorem* : 4 (*Rule 2*)**

If $z = f(x,y)$ and $x = x(u,v)$, $y = y(u,v)$ are all differentiable functions, then :

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \text{and,} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Once again, similar statements hold for functions of three or more variables, in fact if, $w = f(x,y,z,\ldots,t)$ is a differentiable function of any number of variables, and each variable, in turn, is differentiable function of any number of variables $x = x(u,v,\ldots,s)$, $y = y(u,v,\ldots,s)$, . . . , $t = t(u,v,\ldots,s)$, then for example,

$$\frac{dw}{du} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \ldots + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}.$$

This is the most general statement of the chain rule **.**

***Example* : 23**

(i) $z = x^2y^3 + x\sin y$, $x = u^2$, $y = uv$

(ii) $z = e^x \ln y$, $x = u^2 - 2\upsilon$, $y = v^2 - 2u$

Solution

(i) $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= (2xy^3 + \sin y)(2u) + (3x^2y^2 + x\cos y)(v)$$

$$= 7u^6v^3 + 2u\sin uv + u^2v\cos uv$$

$$\frac{\partial z}{\partial \upsilon} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \upsilon} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \upsilon}$$

$$= (2xy^3 + \sin y)(0) + (3x^2y^2 + x\cos y)(u)$$

$$= 3u^7v^2 + u^3\cos uv$$

(ii) $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^{x}\ln y\,(2u) + \frac{e^{x}}{y}(-2)$$

$$= e^{u^2-2v}\left(2u\,\ln(v^2-2u) - \frac{2}{v^2-2u}\right)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \upsilon} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \upsilon} = e^{x}\ln y\,(-2) + \frac{e^{x}}{y}(2u)$$

$$= e^{u^2-2v}\left(-2\,\ln(v^2-2u) + \frac{2u}{v^2-2u}\right)$$

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Example : 24

If $w = r^2 + sv + t^3$ and $r = x^2 + y^2 + z^2$, $s = xyz$, $v = x\,e^{y}$ and $t = y\,z^2$. Find $\frac{\partial w}{\partial z}$.

Solution

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$$

$$= (2r)(2z) + v(xy) + s(0) + 3t^2(2yz)$$

$$= 4z(x^2 + y^2 + z^2) + x^2 y e^{y} + 6y^3 z^5 .$$

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Example : 25

If $w = e^{(x^2+y^2)}$, $x = r\cos\theta$, $y = r\sin\theta$. Show that:

$$\frac{1}{r^2}\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2 = 4e^{2r^2} .$$

Solution

$$\frac{\partial w}{\partial r} = 2re^{r^2}\cos^2\theta + 2re^{r^2}\sin^2\theta = 2re^{r^2} ,$$

$$\frac{\partial w}{\partial \theta} = -2re^{r^2}\sin\theta\cos\theta + 2re^{r^2}\sin\theta\cos\theta = 0 ,$$

$$\frac{1}{r^2}\left(\frac{\partial w}{\partial r}\right)^2 = 4e^{2r^2} , \qquad \left(\frac{\partial w}{\partial \theta}\right)^2 = 0$$

Then : L.H.S. $= \frac{1}{r^2}\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2 = 4e^{2r^2} =$ R.H.S.

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***Example* : 26**

For a differentiable function $f(x, y)$ with $x = r\cos\theta$ and $x = r\sin\theta$, where f_{xy} and f_{yx} are continuous, show that :

$f_r = f_x \cos\theta + f_y \sin\theta$

$f_{rr} = f_{xx} \cos^2\theta + f_{xy} \cos\theta \sin\theta + f_{yy} \sin^2\theta$

Solution

First, notice that, $\dfrac{\partial x}{\partial r} = \cos\theta$ and $\dfrac{\partial y}{\partial r} = \sin\theta$.

$$f_r = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta \quad , \quad \text{Now,}$$

$$f_{rr} = \frac{\partial f_r}{\partial r} = \frac{\partial}{\partial r}\left(f_x \cos\theta + f_y \sin\theta\right)$$

$$= \left[\frac{\partial}{\partial x}(f_x)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(f_x)\frac{\partial y}{\partial r}\right]\cos\theta + \left[\frac{\partial}{\partial x}(f_y)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(f_y)\frac{\partial y}{\partial r}\right]\sin\theta$$

$$= \left(f_{xx} \cos\theta + f_{xy} \sin\theta\right)\cos\theta + \left(f_{yx} \cos\theta + f_{yy} \sin\theta\right)\sin\theta$$

$$= f_{xx} \cos^2\theta + f_{xy} \cos\theta \sin\theta + f_{yy} \sin^2\theta$$

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6-2 Implicit Differentiation

Partial derivatives can be used to find derivatives of functions that are determined implicitly. Suppose, an equation F(x, y) = 0 determines a differentiable function *f* such that $y = f(x)$ that is, $F(x, f(x)) = 0$ for every *x* in the domain of f. Let us introduce the following composite function F: $w = F(u, y)$ where $u = x$ and $y = f(x)$

Using the first rule of chain rules and the fact that *u* and *y* are functions of *one* variable *x* yields, $\dfrac{dw}{dx} = \dfrac{\partial w}{\partial u}\dfrac{du}{dx} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dx}$

Since $w = F(x, f(x)) = 0$ for every x, it follows that $dw/dx = 0$. Moreover, since *u* = *x* and *y* $y = f(x)$, $\dfrac{du}{dx} = 1$, and $\dfrac{dy}{dx} = f'(x)$.

Substituting in the preceding chain rule formula for dw/ dx, we obtain,

$$0 = \frac{\partial w}{\partial u}(1) + \frac{\partial w}{\partial y} f'(x)$$

If $\partial w/\partial y \neq 0$, then (since *u = x)*,

$$f'(x) = \frac{dy}{dx} = -\frac{\partial w/\partial u}{\partial w/\partial y} = -\frac{\partial w/\partial x}{\partial w/\partial y} = -\frac{F_x(x, y)}{F_y(x, y)}$$

We may summarize the preceding discussion as follows.

***Theorem* : 5**

If an equation $F(x, y) = 0$ determines, implicitly. a differentiable function f of one variable *x* such that $y = f(x)$, then, $\dfrac{dy}{dx} = -\dfrac{F_x(x,y)}{F_y(x,y)}$

***Example* : 27**

Find dy/dx if $y = f(x)$ is determined implicitly by,

(i) $y^4 + 3y - 4x^3 - 5x - 1 = 0$, (ii) $y^3 - 3xy = 5x^2 y^2$

Solution

(i) $F(x, y) = y^4 + 3y - 4x^3 - 5x - 1$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{-12x^2 - 5}{4y^3 + 3} = \frac{12x^2 + 5}{4y^3 + 3}$$

(ii) $F(x, y) = y^3 - 3xy - 5x^2 y^2$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{-3y - 10xy^2}{3y^2 - 3x - 10x^2 y} = \frac{3y + 10xy^2}{3y^2 - 3x - 10x^2 y}$$

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In analogy with the single-variable case, we say that the function $z = f(x, y)$ of two variables *x* and *y* is determined *implicitly* as follows.

* Define the composite function $F(x, y, f(x, y)) = 0$ as,

$w = F(u, v, z)$ where $u = x$, $v = y$, $z = f(x, y)$

** Apply the second rule of chain rules,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$$

Which may be written as, $0 = \dfrac{\partial w}{\partial x}(1) + \dfrac{\partial w}{\partial y}(0) + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial x}$

and if $\partial w/\partial z \neq 0$, then , $\dfrac{\partial z}{\partial x} = -\dfrac{\partial w/\partial x}{\partial w/\partial z} = -\dfrac{F_x(x,y,z)}{F_z(x,y,z)}$

The formula for $\partial w/\partial y$ may be obtained in similar fashion.

Theorem : 6

If an equation $F(x, y, z) = 0$ determines, implicitly. a differentiable function f of two variable x and y such that $z = f(x, y)$ then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

Example : 28

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = f(x, y)$ is determined implicitly by,

$$x^2 z^2 + x y^2 - z^3 + 4yz - 5 = 0$$

Solution

$$F(x,y,z) = x^2 z^2 + x y^2 - z^3 + 4yz - 5$$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{2xz^2 + y^2}{2x^2 z - 3z^2 + 4y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{2xy + 4z}{2x^2 z - 3z^2 + 4y}$$

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Exercise (3– 5)

(I) Find $\frac{dw}{dt}$ for the following functions,

(1) $w = x^2 y - 3y^2$; $x = 3t + 1,\quad y = t^2$

(2) $w = \ln(x^2 + y)$; $x = \sqrt{t}\ ,\quad y = \sqrt[3]{t^2}$

(3) $w = e^x\ y$; $x = \ln\ t\ ,\quad y = 3t^2$

(4) $w = 3\cos x - \sin\ xy$; $x = 1/t\ ,\quad y = 3t$

(5) $w = \tan^{-1}\ xy$; $x = \tan t\ ,\quad y = 1/t$

(6) $w = \sqrt{1 + y + 3xy^2\ z}$; $x = \ln t,\quad y = t^2,\ z = 2t$

(7) $w = x^2 y + e^{2x - y}$ $x = \cos t\ ,\quad y = 4t^3$

(8) $w = x\sin y + y\cos z + e^z$; $x = t,\quad y = t^2,\ z = \ln t$

(9) $w = \sin(xyz)$, $x = 1/t\ ,\ y = \ln t\ ,\ z = t$

(10) $w = \sin^{-1}\ xy$; $x = \sin t\ ,\quad y = 1/t$

(11) $w = \sqrt{x^2 + y^2 + z^2}$; $x = \cos\ t\ ,\quad y = \sin t,\ z = t$

(II) Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ for the following functions,

(1) $z = x^2 y^3 + x\sin y$; $x = u^2,\quad y = uv$

(2) $z = x^3 + x\ y + y^3$; $x = 2u,\quad y = 3v$

(3) $z = e^x\ \ln y$; $x = u^2 - 2v,\quad y = v^2 - 2u$

(III) Find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$, and $\frac{\partial w}{\partial t}$ for the following functions

(1) $w = x^2 y^3 z + xyz^2$; $x = r^2 + s^2,\ y = rst,\ z = r + s + t$

(2) $w = e^{xyz} + \ln(x + y + z)$; $x = r - s^2,\ y = r + t,\ z = rst$

(3) $w = x^2 + y^2 + z^2$ $x = t,\ y = s,\ z = r$

(IV) For a differentiable function $f(x, y)$ with $x = r\cos\theta$ and $x = r\sin\theta$ and where

f_{xy} and f_{yx} are continuous, show that :

(1) $f_\theta = -f_x\, r\sin\theta + f_y\, r\cos\theta$

(2) $f_{\theta\theta} = f_{xx}\, r^2\sin^2\theta - 2f_{xy}\, r^2\cos\theta\sin\theta + f_{yy}\, r^2\cos^2\theta - f_x\, r\cos\theta - f_y\, r\sin\theta$

(V) Find $\frac{dy}{dx}$ if $y = f(x)$ is determined implicitly by the following equations

(1) $x^2 + 6xy = 5y^2 - 3$

(2) $x^3 - 2xy^2 + y^3 = \sin x$

(3) $e^x = \tan y$

(4) $\ln x^3 + e^{xy} = \sinh x$

(5) $x\sin y - y\cos x = 0$

(6) $xe^y + ye^x = x^2 y$

(VI) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = f(x, y)$ is determined implicitly by the following equations

(1) $4z^4 = 2xy^2 - 3z^2y$

(2) $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$

(3) $xe^{yz} + ye^{xz} = z$

(4) $yx^2 + z^2y + \cos xyz = 4$

(VII) (a) Show that $z = e^{(x+2y)} + \ln(x^2 + 4xy + 4y^2)$, satisfies the equation:

$$\frac{\partial z}{\partial y} = 2\frac{\partial z}{\partial x}$$

(b) Show that any differential function *z = f(x,2y)* will satisfy the previous equation.

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7- Directional Derivatives And The Gradient

7-1 Directional Derivative

The derivative of a function $f(x,y)$ at a point $P(x_0, y_0)$ in the direction of a unit vector $u = u_1 i + u_2 j$ is defined by,

$$D_u f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + t u_1, y_0 + t u_2) - f(x_0, y_0)}{t},$$

provided the limit exist.

This derivative is called the directional derivative of the function $f(x,y)$ at a point $P(x_0, y_0)$ in the direction a unit vector $u = u_1 i + u_2 j$.

Now we may consider the partial derivative $\partial f / \partial x$ as the directional derivative in the direction of *x-axis* ($u = 1i + 0j$) while the partial derivative $\partial f / \partial y$ as the directional derivative in the direction of *y-axis* ($u = 0i + 1j$).

The following theorem provides a formula for finding directional derivatives.

***Theorem* : 7**

> If f is differentiable function of two variables and $u = u_1 i + u_2 j$ is a unit vector, then : $D_u f(x,y) = f_x(x,y) u_1 + f_y(x,y) u_2$.

***Example* : 29**

Find the derivative of the function $f(x,y) = x^4 y^3 + x^3 y$ at the point $P(2, -1)$ in the direction of the vector $a = i + 2j$.

Solution

The vector a is not a unit vector. The unit vector in the direction of the vector a is

$$u = a / \| a \| \quad \Rightarrow. \quad u = \frac{i + 2j}{\sqrt{1+4}} = \frac{1}{\sqrt{5}} i + \frac{2}{\sqrt{5}} j.$$

$$f_x(x, y) = 4x^3 y^3 + 3x^2 y, \qquad f_x(2, -1) = -44$$

$$f_y(x, y) = 3x^4 y^2 + x^3, \qquad f_y(2, -1) = 56.$$

$$D_u f(2, -1) = (-44)\frac{1}{\sqrt{5}} + (56)\frac{2}{\sqrt{5}} = \frac{68}{\sqrt{5}}.$$

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Example **: 30**

Find the derivative of the function $f(x,y) = x^2 + xy$ at the point $P(1, 2)$ in the direction of the vector $a = (1/\sqrt{2})i + (1/\sqrt{2})j$.

Solution

The vector a is a unit vector.

$f_x(x, y) = 2x + y \quad , \quad f_x(1, 2) = 4$

$f_y(x, y) = x \quad , \quad f_y(1, 2) = 1.$

$$D_u f(1,) = (4)\frac{1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

For the function of three variables $f(x,y,z)$, the directional derivative of f at a point $P(x,y,z)$ in the direction of a unit vector $u = u_1 i + u_2 j + u_3 k$,

$$D_u f(x,y,z) = f_x(x,y,z)u_1 + f_y(x,y,z)u_2 + f_z(x,y,z)u_3$$

Example **: 31**

Find the derivative of the function $f(x,y,z) = x^2 z^3 + y^3 z$ at the point $P(1, 1, 1)$ in the direction of the vector $a = i + 2j + 2k$.

Solution

The vector a is not a unit vector. The unit vector in the direction of the vector a is

$$u = a/\|a\| \Rightarrow \quad u = \frac{i + 2j + 2k}{\sqrt{1 + 4 + 4}} = \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k.$$

$f_x(x, y,z) = 2xz^3 \quad , \quad f_x(1, 1,1) = 2$

$f_y(x, y,z) = 3y^2 z \quad , \quad f_y(1,1,1) = 3.$

$f_z(x, y,z) = 3x^2 z^2 + y^3 \quad , \quad f_z(1, 1,1) = 4$

$$D_u f(1,1, 1) = (2)\frac{1}{3} + (3)\frac{2}{3} + (4)\frac{2}{3} = \frac{16}{3} = 5.3333333$$

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7-2 The Gradient

We may express a directional derivative as a dot product of two vectors, as follows:

$$D_u f(x,y) = \left(f_x(x,y)\ i + f_y(x,y)\ j\right) \bullet \left(u_1 i + u_2 j\right)$$

or $$D_u f(x,y) = \left(f_x(x,y),\ f_y(x,y)\right) \bullet \left(u_1,\ u_2\right)$$

The vector in the first bracket, whose components are the first partial derivatives of $f(x,y)$, is very important. It is denoted by $\nabla f(x,y)$ and is given the following special name.

***Definition* : 6**

Let f be a function of three variables. The **gradient** of $f(x,y,z)$ is the vector function given by, $\nabla f(x,y,z) = f_x(x,y,z)\ i + f_y(x,y,z)\ j + f_z(x,y,z)\ k$

Now the directional derivative may be written as a dot product of the gradient vector $\nabla f(x,y)$ and direction unit vector $u = (u_1,\ u_2)$ as,

$$D_u f(x,y,z) = \nabla f(x,y,z) \bullet u$$

***Example* : 32**

Find the directional derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P(1, 1, 0)$ in the direction of $a = 2i - 3j + 6k$.

Solution

The unit vector in the direction of a is, $u = \dfrac{2i - 3j + 6k}{\sqrt{4 + 9 + 36}} = \dfrac{2}{7}i - \dfrac{3}{7}j + \dfrac{6}{7}k$

The gradient of f at P is, $\nabla f(x,y,z) = \left(3x^2 - y^2,\ -2xy,\ -1\right)$

$$\nabla f(1,1,0) = (2,\ -2,\ -1)$$

The directional derivative of f at P in the direction of a is therefore,

$$D_u f(1,1,0) = (2,\ -2,\ -1) \bullet \left(\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}\right) = \frac{4}{7}$$

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<u>Exercise</u> (3– 6)

(I) Find the directional derivatives at point *p* in the indicated directions.

(1) $f(x, y) = x^2 + y^2$, P(1, 1), $a = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

(2) $f(x, y) = 2xy - 3y^2$, P(5, 5), $a = 4i + 3j$

(3) $f(x, y) = e^{xy}$, P(−1, 3), $a = \frac{\sqrt{3}}{2} i + \frac{1}{2} j$

(4) $f(x, y) = \tan^{-1} y/x$, P(4, −4), $a = 2i - 3j$

(5) $f(x, y) = xy + \tan y$, P(−1, π/4), $a = i - 3j$

(6) $f(x, y) = x^2 \ln y$, P(5, 1), $a = -i + 4j$

(7) $f(x, y, z) = xy + z^2$, P(–2, 1, 3), $a = i - 2j + \sqrt{5}\, k$

(8) $f(x, y, z) = xy^3z^2$, P(2, −1, 4), $a = i + 2j - 3k$

(9) $f(x, y, z) = 40 - xyz$, P(3, 0, 1), from P to Q(1, 1, 1)

(10) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, P(–2, 3, 1), from P to Q(0, -5, 4)

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8- Applications On Derivatives

If $r = g(t)i + h(t)j + k(t)k$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function f, then $f(g(t),h(t),k(t)) = c$. Differentiating both sides of this equation with respect to t leads to,

$$\frac{d}{dt}\big(f(g(t),h(t),k(t))\big) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right)}_{\nabla f} \bullet \underbrace{\left(\frac{dg}{dt} i + \frac{dh}{dt} j + \frac{dk}{dt} k\right)}_{dr/dt} = 0$$

At every point along the curve, ∇f is orthogonal to the curve's tangent vectors.

8-1 Tangent Plane And Normal Vector

Now let us restrict our attention to the curves that pass through P . All the tangent vectors at P are orthogonal to ∇f at P, so the curves' tangent lines all lie in the plane through P normal to ∇f. We call this plane the tangent plane of the surface at *P*. The line through *P* perpendicular to the plane is the surface's normal line at P.

***Definition* : 7**

The tangent plane at the point $P(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ is the plane through *P* normal to $\nabla f\big|_p$.
The normal line of the surface at P is the line through P parallel to $\nabla f\big|_p$.

Thus, the equation of the tangent plane is , $\big(\nabla f(x_0, y_0, z_0)\big) \bullet \big((x - x_0, y - y_0, z - z_0)\big) = 0$

Or

$$f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

and the equation of the normal vector,

$$\frac{f_x(x_0,y_0,z_0)}{(x - x_0)} = \frac{f_y(x_0,y_0,z_0)}{(y - y_0)} = \frac{f_z(x_0,y_0,z_0)}{(z - z_0)}$$

***Example* : 33**

Find the equations of tangent plane and normal line of the surface $x^2 + y^2 + z - 9 = 0$ at the point $P(1, 2, 4)$.

Solution

$$f(x, y, z) = x^2 + y^2 + z - 9$$

$$\nabla f(x, y, z) = (2x, 2y, 1), \quad \nabla f(1, 2, 4) = (2, 4, 1)$$

Then, the equation of the tangent plane :

$$f_x(x_0, y_0, z_0)(x - x_0) + f_x(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

i.e. $2(x - 1) + 4(y - 2) + 1(z - 4) = 0$, or $2x + 4y + z = 14$

and the equation of the normal line,

$$\frac{f_x(x_0, y_0, z_0)}{(x - x_0)} = \frac{f_y(x_0, y_0, z_0)}{(y - y_0)} = \frac{f_z(x_0, y_0, z_0)}{(z - z_0)}$$

i.e. $\dfrac{2}{x - 1} = \dfrac{4}{y - 2} = \dfrac{1}{z - 4}$

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***Example* : 34**

Find the equations of tangent plane and normal line to the ellipsoid $\frac{3}{4}x^2 + 3y^2 + z^2 = 12$ at the point $P(2, 1, \sqrt{6})$.

Solution

$$f(x, y, z) = \frac{3}{4}x^2 + 3y^2 + z^2 - 12$$

$$\nabla f(x, y, z) = \left(\frac{3}{2}x, 6y, 2z\right), \qquad \nabla f(2, 1, \sqrt{6}) = (3, 6, 2\sqrt{6})$$

Then, the equation of the tangent plane ,

$$3(x - 2) + 6(y - 1) + 2\sqrt{6}(z - \sqrt{6}) = 0, \quad \text{or} \quad 3x + 6y + 2\sqrt{6}\,z = 24$$

and the equation of the normal line, $\dfrac{3}{x - 2} = \dfrac{6}{y - 1} = \dfrac{2\sqrt{6}}{z - \sqrt{6}}$

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8-2 Extrema Values And Saddle points

To find the local extreme values of a function of a single variable, we look for points where the graph has a horizontal tangent line. At such points we then look for local maxima, local minima, and points of inflection. For a function $f(x,y)$ of two variables, we look for points where the surface $z = f(x,y)$) has a horizontal tangent plane. At such points we then look for **local maxima**, **local minima**, and **saddle points**.

***Definition* : 8**

Let $f(x,y)$ be defined on a region *R* containing the point (a, b). Then

1. $f(a,b)$ is a **local maximum** value of f if $f(a,b) \ge f(x,y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a,b)$ is a **local minimum** value of f if $f(a,b) \le f(x,y)$ for all domain points (x, y) in an open disk centered at (a, b).
3. $f(a,b)$ is a **saddle point** value of f if there are domain points (x, y) where $f(x,y) > f(a,b)$ and domain points (x, y) where $f(x,y) < f(a,b)$ in every open disk centered at (a, b).

As with functions of a single variable, the key to identifying the local extrema is a first derivative test.

Theorem : 8

If $f(x,y)$ has a **local maximum**, **local minimum** or **saddle point** value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$

***Definition* : 9**

An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a critical point of f.

Thus, the only points where a function $f(x,y)$ can assume extreme or saddle points values are critical points and boundary points.

The fact that $f_x = f_y = 0$, at an interior point (a, b) of *R* does not tell us enough to be sure f has a local extreme value there. However, if f and its first and second partial derivatives are continuous on *R*, we may be able to learn the rest from the following theorem. Define the function: $F(x, y) = f_{xx} f_{yy} - f_{xy}^2$ which is called the discriminate of f. It is sometimes easier to remember the function $F(x,y)$ determinant form,

$$F(x, y) = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$$

***Theorem* : 9**

Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b)) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if : $f_{xx} < 0$ and $F(x,y) > 0$ at (a, b)

ii) f has a **local minimum** at (a, b) if : $f_{xx} > 0$ and $F(x,y) > 0$ at (a, b)

iii) f has a **saddle point** at (a, b) if : $F(x,y) < 0$ at (a, b)

iv) The test is failed if : $F(x,y) = 0$ at (a, b)

***Example* : 35**

Find the local extreme values of $f(x,y) = xy$.

***Solution*.**

The function is defined and differentiable for all *x* and *y* and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to, $f_x = y = 0$ and $f_y = x = 0$

Thus, the origin is the only point where *f* might have an extreme value. To see what happens there, we calculate

$$f_{xx} = 0, \qquad f_{yy} = 0, \qquad f_{xy} = 1$$

$$F(x, y) = f_{xx} f_{yy} - f_{xy}^2 = -1 < 0$$

Therefore the function has a **saddle point** at (0, 0), has no local extreme values.

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Example **: 36**

Find the local extreme values of

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution

Since f is differentiable everywhere, it can assume extreme values only where,

$$f_x = y - 2x - 2 = 0 \quad \text{and} \quad f_y = x - 2y - 2 = 0$$

or $\quad x = y = -2$

Thus, the point $(-2, -2)$ is the only point where *f* may take on an extreme value.

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1$$

$$F(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 3 > 0$$

Then, $f_{xx} = -2 < 0$ and $F(x, y) = 3 > 0$

Therefore the function has a **local maximum** at $(-2, -2)$.

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Example **: 37**

Find the local extreme values of: $f(x,y) = x^2 - 4xy + y^3 + 4y$

Solution

Since *f* is differentiable everywhere, and,

$$f_x = 2x - 4y = 0 \quad \text{and} \quad f_y = -4x + 3y^2 + 4 = 0.$$

Solving this system, we find that *f* has the two critical points $(4,2)$ and $(4/3, 2/3)$. The second partial derivatives are,

$$f_{xx}(x, y) = 2, \quad f_{yy}(x,y) = 6y, \quad f_{xy}(x,y) = -4$$

$$F(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 12y - 16$$

(a,b)	$F(a,b)$	$f_{xx}(a,b)$	Conclusion
$(4,2)$	$8 > 0$	$2 > 0$	$f(4,2)=0$ is a local min.
$(4/3, 2/3)$	$-8 < 0$	-	$f(\frac{4}{3},\frac{2}{3})=1.19$ is a saddle point

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Example : 38

Find the local extreme values of: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Solution

Since *f* is differentiable everywhere, and,

$$f_x = 3x^2 - 3 = 0 \quad \text{and} \quad f_y = 3y^2 - 12 = 0.$$

i.e. $x = \pm 1$ and $y = \pm 2$

We find that *f* has the four critical points,

$$(1,2),\ (-1,\ 2),\ (1,\ -2) \text{ and } (-1,\ -2).$$

The second partial derivatives are,

$$f_{xx}(x,\ y) = 6x, \quad f_{yy}(x,y) = 6y \ , \quad f_{xy}(x,y) = 0$$

$$F(x,\ y) = f_{xx}\, f_{yy} - f_{xy}^2 = 36\, x\, y$$

(a,b)	$F(a,b)$	$f_{xx}(a,b)$	Conclusion
$(1,\ 2)$	$72 > 0$	$6 > 0$	$f(1,2) = 2$ is a local min.
$(-1,\ 2)$	$-72 < 0$	-	$f(-1,2) = 6$ is a saddle point
$(1,\ -2)$	$-72 < 0$	-	$f(1,-2) = 34$ is a saddle point
$(-1,\ -2)$	$72 > 0$	$-6 < 0$	$f(-1,-2) = 38$ is a local max.

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8-3 Constraints And Lagrange Multipliers

We sometimes need to find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane

Here, we explore a powerful method for finding extreme values of constrained functions: the method of ***Lagrange multipliers***.

Theorem : 10 (***Lagrange's Theorem***)

Suppose f and g are functions of two variables that have continuous first partial derivatives, and that $\nabla g \neq 0$ throughout a region of the *xy-plane*. If *f* has an extremum $f(a,b)$ subject to the constraint $g(x,y) = 0$, then there is a real number λ such that, $\nabla f(a,b) = \lambda \nabla g(a,b)$

The points at which a function f of two variables has local extrema subject to the constrain $g(x,y)=0$ are included among the point (x,y) determined by the first two coordinates of the solution (x,y,λ) of the system of equations,

$$f_x(x,y) = \lambda g_x(x,y),\ f_y(x,y) = \lambda g_y(x,y),\ g(x,y)=0.$$

The Lagrange's Theorem (3.8.3) may be extended to the function of three variables x, y, z. In this case, we solve the system,

$$\begin{cases} f_x(x,y,z) = \lambda g_x(x,y,z) \\ f_y(x,y,z) = \lambda g_y(x,y,z) \\ f_z(x,y,z) = \lambda g_z(x,y,z) \\ g(x,y,z) = 0 \end{cases}$$

Some applications may involve more than one constraint. In particular, consider the problem of finding the extrema of $f(x,y,z)$ subject to the two constraints,

$$g(x,y,z) = 0 \text{ and } h(x,y,z) = 0.$$

Then the following condition must be satisfied for some real numbers λ and μ such that,

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

And we solve the system,

$$\begin{cases} f_x(x,y,z) = \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) = \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) = \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{cases}$$

***Example* : 39**

Find the maximum and minimum values of the function: $f(x,y)=3x+4y$ on the circle $x^2+y^2=1$

Solution

Let $f(x,y)=3x+4y$ and $g(x,y)=x^2+y^2-1$

The system,

$$f_x(x,y) = \lambda g_x(x,y),\quad f_y(x,y) = \lambda g_y(x,y),\quad g(x,y)=0.$$

Leads to the equations,

$$3 = 2\lambda x,\ (1) \qquad 4 = 2\lambda y,\ (2) \qquad x^2+y^2=1\ (3)$$

From (1),(2), $x=\dfrac{3}{2\lambda}$, $y=\dfrac{2}{\lambda}$. Substituting into (3), We obtain $\lambda = \pm\frac{5}{2}$ and so, $x = \pm\frac{3}{5}$, $y = \pm\frac{4}{5}$

(x,y)	$(3/5,\ 4/5)$	$(-3/5,\ -4/5)$
$f(x,y)$	5	-5

Then $f\left(\frac{3}{5}, \frac{4}{5}\right) = 5$ is a local maximum.

and $f\left(\frac{-3}{5}, \frac{-4}{5}\right) = -5$ is a local minimum.

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***Example* : 40**

Find the extrema of $f(x,y) = xy$ if (x, y) is restricted to the ellipse $4x^2 + y^2 = 4$.

Solution

Let $f(x,y) = xy$ and $g(x,y) = 4x^2 + y^2 - 4$

The system,

$$f_x(x,y) = \lambda g_x(x,y),\ f_y(x,y) = \lambda g_y(x,y),\ g(x,y) = 0.$$

Leads to the equations,

$$y = 8\lambda x,\ (1) \qquad x = 2\lambda y,\ (2) \qquad 4x^2 + y^2 = 4 \quad (3)$$

From (1),(2), $x = 16x\lambda^2$ or $x(1 - 16\lambda^2) = 0$

Therefore either $x = 0$ or $\lambda = \pm 1/4$

If $x = 0$, then from (3), $y = \pm 2$

If $\lambda = \pm 1/4$, then from (1), $y = 8\lambda x = 8x(\pm 1/4) = \pm 2x$.

Substituting into (3), $8x^2 = 4$ or $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\sqrt{2}$

Now :

$$f(0, \pm 2) = 0,\quad f\left(\frac{\pm 1}{\sqrt{2}}, \pm\sqrt{2}\right) = 1,\quad f\left(\frac{\pm 1}{\sqrt{2}}, \mp\sqrt{2}\right) = -1.$$

Thus, $f(x,y) = xy$ takes on a maximum value of 1 at either $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ or $\left(\frac{-1}{\sqrt{2}}, -\sqrt{2}\right)$

and a minimum value of -1 at $\left(\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ or $\left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right)$

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***Example* : 41**

If $f(x,y,z) = 4x^2 + y^2 + 5z^2$. Find the point on the plane $2x + 3y + 4z = 12$ at which $f(x,y,z)$ has its least value

Solution

Let $f(x,y,z) = 4x^2 + y^2 + 5z^2$ and $g(x,y) = 2x + 3y + 4z - 12$

$\nabla f = (8x, 2y, 10z)$, $\nabla g = (2, 3, 4)$

This leads to the equations,

$8x = 2\lambda$, (1) $2y = 3\lambda$, (2) $10z = 4\lambda$, (3) $2x + 3y + 4z = 12$ (4)

From (1),(2), (3), $\lambda = 4x = \frac{2}{3}y = \frac{5}{2}z$. or $y = 6x$, $z = \frac{8}{5}x$

Substituting into (4) , $2x + 18x + \frac{32}{5}x = 12$

Therefore , $x = \frac{5}{11}$, $y = \frac{30}{11}$, $z = \frac{8}{11}$

Since there is only one critical point, it follows that the minimum value occurs at that point, (5/11,30/11,8/11).

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***Example* : 42**

Find the shortest distance between the origin and the surface: $z = xy + 1$.

Solution

Consider the point (x,y,z) on the surface $z = xy + 1$. The distance between the point (x,y,z) and the origin is, $d = \sqrt{x^2 + y^2 + z^2}$.

Now , we can restate the problem as, find the minimum value of the function

$f(x,y,z) = x^2 + y^2 + z^2$ on the surface $z = xy + 1$.

$f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = z - xy - 1$

$\nabla f = (2x, 2y, 2z)$, $\nabla g = (-y, -x, 1)$

This leads to the equations,

$2x = -\lambda y$ (1) $2y = -\lambda x$ (2) $2z = \lambda$, (3) $z = xy + 1$ (4)

From (1),(2), $(x - y)(\lambda + 2) = 0$, $x = y$ or $\lambda = -2$

Then from (3), (4), $z = 1$ and $x = y = 0$

Thus the shortest distance, $d = \sqrt{x^2 + y^2 + z^2} = 1$.

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Example **: 43**

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

Solution

Consider the point (x, y, z) on the ellipse (the intersection of the above two surfaces). The distance between the point (x, y, z) and the origin is, $d = \sqrt{x^2 + y^2 + z^2}$

We find the extreme values of : $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints:

$$g_1(x, y, z) = x^2 + y^2 - 1, \qquad g_2(x, y, z) = x + y + z - 1$$

The gradient equation then gives

$$\nabla f(x, y, z) = \lambda \nabla g_1(x, y, z) + \mu \nabla g_2(x, y, z)$$

Then, $(2x, 2y, 2z) = \lambda(2x, 2y, 0) + \mu(1,1,1)$

This leads to the equations,

$$2x = 2\lambda x + \mu \quad (1) \quad , \quad 2y = 2\lambda y + \mu \quad (2) \quad , \quad 2z = \mu \quad (3)$$

$$x^2 + y^2 - 1 = 0, \quad (4) \quad , \quad x + y + z - 1 = 0 \quad (5)$$

From (3) in (1), (2), $2x = 2\lambda x + 2z, \quad 2y = 2\lambda y + 2z, \quad (6)$

Equations (6) are satisfied simultaneously if either,

$$\lambda = 1 \quad \text{and} \quad z = 0 \quad \text{or} \quad \lambda \neq 1 \text{ and } x = y = z/(1-\lambda)$$

If $z = 0$, then solving equations (4) and (5) simultaneously to find the corresponding points on the ellipse gives the two points $(1, 0, 0)$ and $(0,1, 0)$.

If $x = y$, then (4) and (5) give: $x^2 + x^2 - 1 = 0,$ and $x + x + z - 1 = 0$

i.e. $\quad x = y = \pm \dfrac{1}{\sqrt{2}} \quad$ and $\quad z = 1 \mp \sqrt{2}$.

Then we have four critical points,

$$P_1 = (1, 0, 0),\ P_2 = (0,1, 0),\ P_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right),\ P_4 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$$

$$d(P_1) = 1, \quad d(P_2) = 1, \quad d(P_3) = 1.0824, \quad d(P_4) = 2.61313$$

Now, it is clear that the closest points to the origin are P_1, P_2 and the farthest point from the origin is P_4

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E x e r c i s e (3– 7)

(I) Find the equations of the tangent plane and the normal line to the given level surfaces at the indicated points.

(1) $x^2+y^2+z^2=4$ at $P(1,1,\sqrt{2})$, (2) $x^2+2xy-y^2+z^2=7$ at $P(1,-1,3)$

(3) $x^2+y^2-z^2=1$ at $P(1,1,1)$, (4) $x^2-xy\ y^2-z=0$ at $P(1,1,-1)$

(5) $x^2+y^2-z^2=0$ at $P(1,1,\sqrt{2})$, (6) $x+y+z=1$ at $P(0,1,0)$

(7) $y=x^2$ at $P(2,4,0)$

(II) Find a vector normal to the given level curves at the indicated points.

(1) $3x^2+y^2=4$ at $(1,1)$, (2) $4x+y^2=1$ at $(-2,3)$

(3) $x^2+y=0$ at $(2,-4)$

(III) Find all the local maxima, local minima and the saddle points of the following functions

(1) $f(x,y)=x^2-3xy-y^2+2y-6$, (2) $f(x,y)=x^2+xy+3x+2y+5$

(3) $f(x,y)=x^4+y^3+32x-9y$, (4) $f(x,y)=x^3-y^3-2xy+6$

(5) $f(x,y)=\cos x+\cos y$, (6) $f(x,y)=\sin x+\sin y$

(7) $f(x,y)=e^x\sin y$, (8) $f(x,y)=e^{2x}\cos y$

(IV) Use Lagrange multipliers to find the local extrema for the following functions under the stated constraints

(1) $f(x,y)=y^2-4xy+4x^2,\quad x^2+y^2=1$

(2) $f(x,y)=2x^2+xy-y^2+y,\quad 2x+3y=1$

(3) $f(x,y,z)=x+2y-3z,\quad z=4x^2+y^2$

(4) $f(x,y,z)=x+y+z,\quad x^2+y^2+z^2=25$

(5) $f(x,y,z)=xyz,\quad x^2+4y^2+2z^2=4$

(6) $f(x,y,z)=x^2+y^2+z^2,\quad x-y+z=1$

(7) $f(x,y,z)=z-x^2-y^2,\quad x+y+z=1,\ x^2+y^2=4$

(8) $f(x,y,z)=x^2+y^2+z^2,\quad x-y=1,\ y^2-z^2=1$

(V) Find the point on the sphere: $x^2 + y^2 + z^2 = 9$ that is close to the point: (2, 3, 4).

(VI) Find the point on the space where the sum of whose coordinates is 64 and whose distance from the origin is minimum.

(VII) À rectangular parallelepiped, with sides parallel to the coordinates axis and inscribes in the ellipsoid: $16x^2 + 4y^2 + 9z^2 - 144=0$, What dimension yield the largest volume.

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CHAPTER 4

INFINITE SERIES

Series, in particular power series, play an important rule in mathematics. To introduce the series, we begin with definition of a sequence and related concepts.

1- Infinite Sequences

***Definition* : 1**

An infinite sequence (or sequence) of numbers is a function whose domain is the set of integers greater than or equal to some integer.

Thus a sequence is a set of numbers $u_1, u_2, u_3, \ldots,$ in a definite order of arrangement and formed according to a definite rule. Each number in the sequence is called a term, the term u_n is called the nth term. The sequence is called finite or infinite according as there is or is not a finite number of terms. In this section we shall consider the infinite sequences only. An infinite sequence or, briefly, a sequence is denoted by $\{u_n\}$. In this chapter, the range of the sequence will be a set of real numbers.

The graph of the sequence may be represented as a set of points (n, u_n) in the *xy*-plane.

***Example* : 1**

Represent the following sequences,

$(i)\ \left\{\dfrac{(-1)^{n+1}}{n}\right\}$ $(ii)\ \left\{\dfrac{n-1}{n}\right\}$ $(iii)\ \{3\}$

$(\text{iv})\ \left\{\dfrac{(-1)^{n+1}(n-1)}{n}\right\}$ $(\text{v})\ \{n-1\}$ $(\text{vi})\ \left\{2+(0.1)^{n}\right\}$

Solution

$$\text{(i)}\ \left\{\frac{(-1)^{n+1}}{n}\right\} = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots\ .$$

$$\text{(ii)}\ \left\{\frac{n-1}{n}\right\} = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ .$$

$$\text{(iii)}\ \{3\} = 3, 3, 3, 3, 3, \ldots\ .$$

$$\text{(iv)}\ \left\{\frac{(-1)^{n+1}\ (n-1)}{n}\right\} = 0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots$$

$$\text{(v)}\ \{n-1\} = 0, 1, 2, 3, 4, \ldots .$$

$$(vi)\ \{2 + (0.1)^n\} = 2.1, 2.01, 2.001, 2.0001, 2.00001, \ldots .$$

The sequences (i)-(vi) are illustrated in Fig. (4.1),

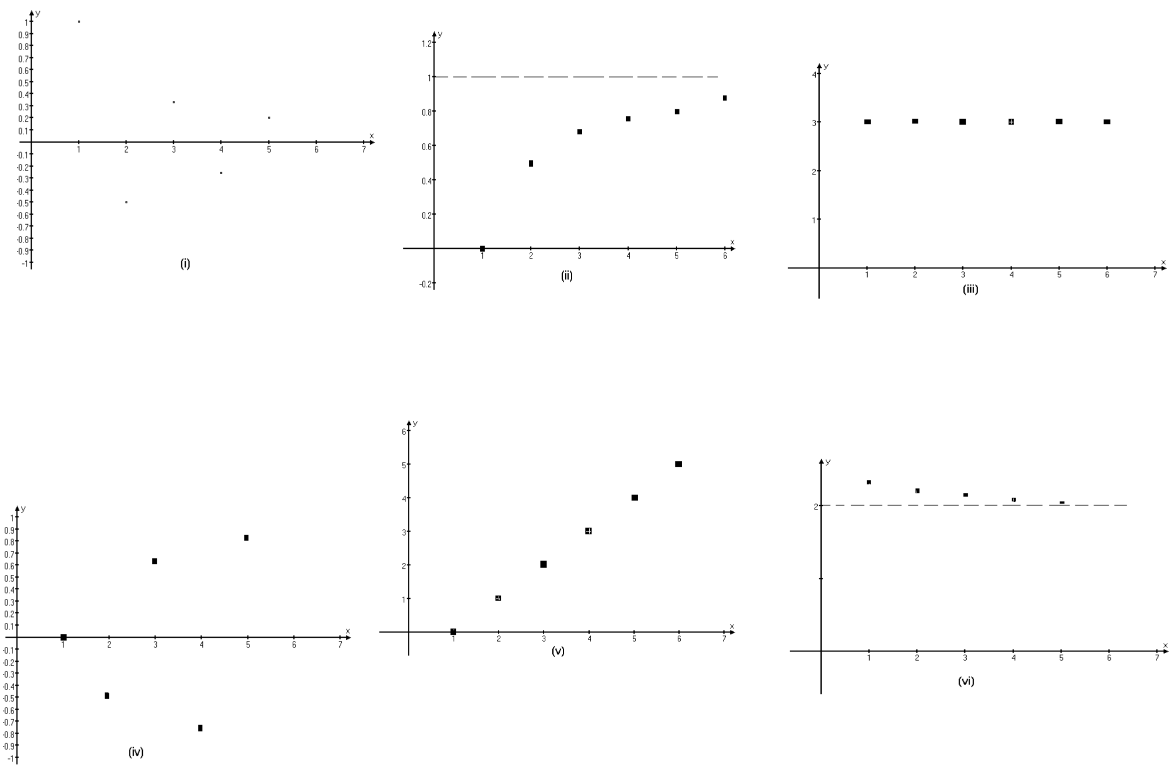

Fig. (4.1)

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In the above examples, we see that as *n* increases, some of these sequences approaches certain values (i, ii, iii, vi) and the others are not (iv, v). This leads us to the following definition.

***Definition* : 2**

A sequence $\{u_n\}$ has the **limit L** or **converges to L** denoted by either $\lim_{n \to \infty} u_n = L$ or $u_n \to L$ as $n \to \infty$, if for every $\varepsilon > 0$ there exist a positive number N such that $|a_n - L| < \varepsilon$, whenever $n > N$.
If such a number L does not exist, the sequence has **no limit** or **diverges.**

N O T E

In example (1) above, sequence (i), (ii), (iii), (vi) **are converge** to 1, 0, 3, 2 respectively while the sequences in (iv) and (v) are diverge.

The next theorem is important because' it allows us to use results from limit of function of one variable to investigate convergence or divergence of sequences.

***Theorem* : 1**

Let $\{u_n\}$ be a sequence. Let $f(n) = u_n$, and suppose that $f(x)$ exists for every real number $x \geq 1$.

(i) If $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} f(n) = L$ and $\{u_n\}$ converges to L.

(ii) If $\lim_{x \to \infty} f(x) = \pm\infty$, then $\lim_{n \to \infty} f(n) = \pm\infty$ and $\{u_n\}$ diverges.

***Example* :2**

Determine whether the sequence $\left\{1 + \frac{1}{n}\right\}$ converges or diverges.

Solution

Let $f(x) = 1 + \frac{1}{x}$ for $x \geq 1$, then, $\lim_{x \to \infty} f(x) = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right) = 1.$

Hence, $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right) = 1.$ Thus, the sequence $\left\{1 + \frac{1}{n}\right\}$ converges to 1.

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***Example* :3**

Determine whether the following sequences converges or diverges

(a) $\left\{\dfrac{n^3}{3} - 4\right\}$ (b) $\left\{(-1)^{n+1}\right\}$ (c) $\left\{5n/e^{2n}\right\}$

Solution

(a) Let $f(x) = \dfrac{x^3}{3} - 4$ for $x \geq 1$, then, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(\dfrac{x^3}{3} - 4\right) = \infty$

Hence the sequence $\left\{\dfrac{n^3}{3} - 4\right\}$ diverges.

(b) The sequence $\left\{(-1)^{n+1}\right\} = 1, -1, 1, -1, 1, \ldots.$ We see that the terms of the sequence oscillate between 1 and -1 . Thus $\lim\limits_{n\to\infty}(-1)^{n+1}$ does not exist and the sequence $\left\{(-1)^{n+1}\right\}$ diverges.

(c) Let $f(x) = 5x/e^{2x}$ for $x \geq 1$, then using L'Hopital's rule we obtain,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{5x}{e^{2x}}\right) = \lim_{x\to\infty}\left(\frac{5}{2e^{2x}}\right) = 0.$$

Then $\lim\limits_{n\to\infty}\left(\dfrac{5n}{e^{2n}}\right) = 0$, and the sequence $\left\{\dfrac{5n}{e^{2n}}\right\}$ converges to 0.

Note that all theorems of the limit can apply to the function $f(n)$ as $n \to \infty$ to evaluate $\lim\limits_{n\to\infty} f(n)$ directly.

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***Example* :4**

Determine whether the following sequences converges or diverges

(a) $\left\{\dfrac{3n^2 - 5n}{2n^2 + n - 6}\right\}$ (b) $\left\{\dfrac{3n^2 + 2n}{2n - 1}\right\}$ (c) $\left\{\dfrac{1 + 2.10^n}{4 + 3.10^n}\right\}$

Solution

(a) $\lim\limits_{n\to\infty}\dfrac{3n^2 - 5n}{2n^2 + n - 6} = \lim\limits_{n\to\infty}\dfrac{3 - 5/n}{2 + 1/n - 6/n^2} = \dfrac{3}{2}.$

Thus the sequence $\left\{\dfrac{3n^2 - 5n}{2n^2 + n - 6}\right\}$ converges to $\dfrac{3}{2}$.

(b) $\lim\limits_{n\to\infty}\dfrac{3n^2 + 2n}{2n - 1} = \lim\limits_{n\to\infty}\dfrac{3 + 2/n}{2/n - 1/n^2} = \infty.$

Thus the sequence $\left\{\dfrac{3n^2 + 2n}{2n - 1}\right\}$ diverges.

(c) $\lim\limits_{n\to\infty} \dfrac{1 + 2.10^n}{4 + 3.10^n} = \lim\limits_{n\to\infty} \dfrac{10^{-n} + 2}{4.10^{-n} + 3} = \dfrac{2}{3}$.

Thus the sequence $\left\{\dfrac{1 + 2.10^n}{4 + 3.10^n}\right\}$ converges to $\dfrac{2}{3}$.

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Definition **: 3**

A sequence $\{u_n\}$ is **bounded** if there is a real positive number M such that : $|u_k| \le M$, for every k.

Example **: 5**

The sequence $\left\{\dfrac{n}{n+1}\right\}$ has the terms $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}, \ldots,$ then for every *n*, $|u_k| \le 1$. Thus for any positive real number $M \ge 1$, the sequence $\left\{\dfrac{n}{n+1}\right\}$ is bounded.

Definition **: 4**

If $u_{k+1} \ge u_k$ for every k, the sequence $\{u_n\}$ is called **monotonic increasing**, while if $u_{k+1} > u_k$ for every *k*, it is called **strictly increasing**. Similarly, if, $u_{k+1} \le u_k$ for every k, the sequence is called **monotonic decreasing**, while if $u_{k+1} < u_k$ for every k, it is called **strictly decreasing**

Theorem **: 2**

Every bounded monotonic sequence has a

N O T E

The sequence in example (5) is monotonic increasing and is bounded, so it is converges. It is easy to prove the following theorem,

<u>Theorem</u> **: 3**

$$\lim_{n\to\infty} \left| r^n \right| = \begin{cases} 0 & \text{if } |r| < 1 \\ \infty & \text{if } |r| > 1 \end{cases}$$

<u>Example</u> **: 6**

Determine whether the following sequences converges or diverges

(a) $\left\{ \left(\dfrac{-2}{3} \right)^n \right\}$, (b) $\left\{ (1.03)^n \right\}$

<u>Solution</u>

(a) Since $\left| \dfrac{-2}{3} \right| = \dfrac{2}{3} \prec 1$, then $\lim\limits_{n\to\infty} \left(\dfrac{-2}{3} \right)^n = 0$, and $\left\{ \left(\dfrac{-2}{3} \right)^n \right\}$ converges to 0.

(b) Since $1.03 < 1$, then $\lim\limits_{n\to\infty} (1.03)^n = \infty$, and $\left\{ (1.03)^n \right\}$ diverges.

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Exercise (4–1)

Determine whether the sequence $\{u_n\}$ converges or diverges where u_n has the following expression:

(1) $\dfrac{n}{3n+2}$, (2) $\dfrac{n}{n-1}$, (3) $1+(-1)^{n+1}$

(4) $\dfrac{n^2-2n+1}{n-1}$, (5) $n+7$, (6) $\dfrac{1-2n}{1+2n}$

(7) $6\left(\dfrac{-5}{6}\right)^n$, (8) $\dfrac{n+(-1)^n}{n}$, (9) $8-\left(\dfrac{7}{8}\right)^n$

(10) $\sqrt{\dfrac{2n}{n+1}}$, (11) $\left(1+\dfrac{1}{n}\right)^n$, (12) $\dfrac{\sin n}{n}$

(13) $\dfrac{\cos n}{n}$, (14) $n\sin\dfrac{1}{n}$, (15) $(-1)^n\dfrac{\ln n}{n}$

(16) $n^{1/n}$, (17) $\dfrac{n^2}{3^n}$, (18) $\left(1+(0.1)^n\right)$

(19) $\sqrt{n+1}-\sqrt{n}$, (20) $8n+1$, (21) $\left(\dfrac{2n-3}{3n+7}\right)^4$

(22) $\left(1-\dfrac{1}{2^n}\right)$, (23) $\left(e^{-n}\ln n\right)$, (24) $2^{-n}\sin n$

(28) $\dfrac{n+1}{\sqrt{n}}$, (29) $(-1)^{n+1}$, (30) $\sqrt{n^2-n}-n$

(31) $5n/e^{2n}$, (32) $\dfrac{2}{\sqrt{n^2+9}}$, (33) $\left(\dfrac{3n^2-5n}{2n^2+n-6}\right)$

(34) $\left(\dfrac{3n^2+2n}{2n^2-1}\right)$, (35) $\left(\dfrac{3n^2+2n}{2n-1}\right)$, (36) $\left(\dfrac{1+2.10^n}{4+3.10^n}\right)$

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2- Infinite Series

An infinite series (or briefly, a series) is an expression of the form, $a_1 + a_2 + a_3 + \ . \ . \ . \ + a_n + . \ . \ .$, or in summation notation, $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$, where a_n is the ***nth* term** of the series.

We can define a sequence $\{S_n\}$ such that :

$$S_1 = a_1 , \quad S_2 = a_1 + a_2 , \quad S_3 = a_1 + a_2 + a_3,$$

$$S_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ . \ . \ . \ + a_n \ .$$

The sequence $\{S_n\}$ is called the **sequence of partial sums** of the series $\sum_{n=1}^{\infty} a_n$

***Definition* : 5**

A series $\sum_{n=1}^{\infty} a_n$ is convergent (or converges) if its sequence of partial sums $\{S_n\}$ converges, that is, $\lim_{n\to\infty} S_n = L$, then L is the sum of the series and we write:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ . \ . \ . \ + a_n = L,$$

and we say that the series $\sum a_n$ converges to L . If the sequence $\{S_n\}$ is divergent then the series $\sum a_n$ is divergent (or diverges) and it has no sum.

***Example* : 7**

Determine whether the following series convergence or divergence: $\sum \frac{1}{n(n+1)}$

Solution

The *nth* term of this series can be rewritten (using partial fraction) as,

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

The *nth* term of the sequence of partial sums is

$$S_n = a_1 + a_2 + a_3 + \ . \ . \ . \ + a_n$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + . \ . \ . + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

We see that all numbers in S_n except the first and last numbers cancel,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

Thus the series converges to 1.

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Note:

The above series $\sum \frac{1}{n(n+1)}$ is called a **telescoping series**, since S_n causes the terms telescope to $\left(1 - \frac{1}{n+1}\right)$

***Example* : 8**

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}$ converges or diverges.

Solution

The terms of sequence of partial sums are

$S_1 = 1,\ S_2 = 0,\ S_3 = 1,\ S_4 = 0,\ S_5 = 1,\ S_6 = 0,\ \ldots$ the *nth* term may be written as,

$$S_n = \begin{cases} 1 & \text{if } n \text{ odd.} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Since the sequence of partial sums $\{S_n\}$ oscillates between 1 and 0, it follows that $\lim_{n\to\infty} S_n$ does not exist. Hence the series diverges.

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***Example* : 9**

Determine whether the series $\sum_{n=1}^{\infty} n^2$ converges or diverges.

Solution

The terms of sequence of partial sums are

$$S_1 = 1,\quad S_2 = 1 + 4 = 5,\quad S_3 = 1 + 4 + 9 = 14,\ \ldots,$$

$$S_n = 1 + 4 + 9 + \ldots + n^2.$$

The sequence of partial sums grow beyond every number, and S_n is greater than or equal to n^2 at each stage, then $\lim_{n\to\infty} S_n = \infty$, and hence the series diverges.

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2.1 The Geometric Series

One of the most important series occurs frequently in solutions of applied problems is the geometric series

$$\sum_{n=1}^{\infty} a\, r^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

where *a* and *r* are real numbers with $a \neq 0$.

***Theorem* : 4**

The geometric series, $\sum_{n=1}^{\infty} a\, r^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$

(a) Converges and has sum $S = \dfrac{a}{1-r}$ if $|r| \prec 1$.

(b) Diverges if $|r| \geq 1$.

Proof

If $r = 1$, then, $S_n = a + a + a + \ldots + a = na$

$\lim_{n\to\infty} S_n = \infty$, and hence the series is diverges.

If $r = -1$, then, $S_n = \begin{cases} a & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

$\lim_{n\to\infty} S_n = \textit{does not exist}$, and hence the series is diverges.

If $|r| \neq 1$, $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$, multiply both sides by *r*,

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$, subtracting these two equations,

$(1-r)S_n = a(1-r^n)$ or $S_n = \dfrac{a(1-r^n)}{(1-r)}$

Consequently,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1-r} - \lim_{n\to\infty} \frac{a\, r^n}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n$$

By theorem (4.1.3), $\lim_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1, \\ \infty & \text{if } |r| > 1. \end{cases}$

Hence the series converges to $\dfrac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

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<u>Example</u> **: 10**

Discuss the convergence of the series: $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots$.

<u>Solution</u>

This series is a geometric series with : $a = \frac{1}{9}$, $r = \frac{1}{3} < 1$. Then the series converges to the sum $= \frac{a}{1-r} = \frac{1/9}{1-1/3} = \frac{1}{6}$.

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<u>Example</u> **: 11**

Prove that the following series converges and find its sum, $4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$.

<u>Solution</u>

This is a geometric series with : $a = 4$, $r = \frac{-1}{2}$, *i.e.* $|r| < 1$.

Then this series converges to the sum : $\frac{a}{1-r} = \frac{4}{1+1/2} = \frac{8}{3}$.

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<u>Example</u> **: 12**

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1}$. converges or diverges.

<u>Solution</u>

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - \ldots + 1 + \ldots$$

This is a geometric series with : , i.e. $|r| = 1$. Then it is divergent series.

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2-2 The n^{th} Term Test

Theorem **: 5**

If a series $\sum_{n=1}^{\infty} a_n$ is convergent, then, $\lim_{n\to\infty} a_n = 0$.

Proof .

The nth term a_n of the series can be expressed as $a_n = S_n - S_{n-1}$.

If S is the sum of the series $\sum_{n=1}^{\infty} a_n$, then we know that,

$$\lim_{n\to\infty} S_n = S \quad \text{and also,} \quad \lim_{n\to\infty} S_{n-1} = S .$$

Hence, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$

Note that the converse of this theorem is not true, *i.e.* If $\lim_{n\to\infty} a_n = 0$, it does not necessarily follow that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

As a corollary of the above theorem, we obtain the following test for divergent.

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Corollary **: 1** ***(The n^{th} term test)***

For the series $\sum_{n=1}^{\infty} a_n$,

(i) If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then the series is divergent.

(ii) If $\lim_{n\to\infty} a_n = 0$, the series may be convergent or divergent (Test

Example **: 13**

Determine whether each of the following series converges or diverges

(a) $\sum \frac{n}{3n + 10}$, (b) $\sum n^2$, (c) $\sum \frac{n+1}{n}$

(d) $\sum (-1)^{n+1}$, (e) $\sum \frac{1}{\sqrt{n}}$, (f) $\sum \frac{1}{n^2}$

Solution

(a) $\lim_{n\to\infty} \frac{n}{3n+10} = \frac{1}{3} \neq 0$, hence, $\sum \frac{n}{3n+10}$ diverges.

(b) $\lim_{n\to\infty} n^2 = \infty \neq 0$, hence $\sum n^2$ diverges.

(c) $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$, hence $\sum \frac{n+1}{n}$ diverges.

(d) $\lim_{n\to\infty} (-1)^{n+1}$ does not exist, hence $\sum (-1)^{n+1}$ diverges.

(e) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, hence $\sum \frac{1}{\sqrt{n}}$ may be convergent or divergent (test fails).

(f) $\lim_{n\to\infty} \frac{1}{n^2} = 0$, hence $\sum \frac{1}{n^2}$ may be convergent or divergent (test fails).

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The series in parts (e) and (f) of the above example, further investigation is necessary to decide if the series converges or diverges.

Whenever we have two convergent series we can add them, subtract them, and multiply them by constants, to make other convergent series. The next theorem gives these results,

Theorem **: 6**

If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges to *A* and *B* respectively, then,

(i) $\sum_{n=1}^{\infty} (a_n \pm b_n)$ converges to $A \pm B$.

(ii) $\sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n$ converges to $k A$, k is a constant.

Example **: 14**

Show that the series $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{6^n}$ converges. Find its sum.

Solution

$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{6^n} = \sum_{n=0}^{\infty}\left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$$

The first series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = 1, \ r = \frac{1}{2} \prec 1$.

Then it converges to $\frac{1}{1 - 1/2} = 2$.

The second series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a geometric series with $a = 1, \ r = \frac{1}{3} \prec 1$.

Then it converges to $\frac{1}{1 - 1/3} = \frac{3}{2}$.

Hence the series $\sum_{n=0}^{\infty} \frac{3^n - 2^n}{6^n}$ converges to $2 - \frac{3}{2} = \frac{1}{2}$.

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Example **: 15**

Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{8}{3^{n-1}}$. Find the sum if it exists-

Solution

$$\sum_{n=1}^{\infty} \frac{8}{3^{n-1}} = 8 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$$

the series $= 8 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$ is a convergent geometric series $(a = 1, \ r = 1/3 \prec 1)$.

Then: $$\sum_{n=1}^{\infty} \frac{8}{3^{n-1}} = 8 \frac{1}{1 - 1/3} = 8\left(\frac{3}{2}\right) = 12.$$

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Theorem **: 7**

If $\sum_{n=1}^{\infty} a_n$ is a convergent series and $\sum_{n=1}^{\infty} b_n$ is a divergent, then, $\sum_{n=1}^{\infty} (a_n \pm b_n)$ is divergent.

Example : 15

Determine whether the series $\sum_{n=1}^{\infty}\left((-1)^{n+1} + \frac{1}{3^n}\right)$ converges or diverges

Solution

$$\sum_{n=1}^{\infty}\left((-1)^{n+1} + \frac{1}{3^n}\right) = \sum_{n=1}^{\infty}(-1)^{n+1} + \sum_{n=1}^{\infty}\frac{1}{3^n}$$

The first series = $\sum_{n=1}^{\infty}(-1)^{n+1}$ is divergent, while $\sum_{n=1}^{\infty}\frac{1}{3^n}$ is a convergent geometric series

,then the series , $\sum_{n=1}^{\infty}\left((-1)^{n+1} + \frac{1}{3^n}\right)$ is divergent

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Exercise (4– 2)

(I) Determine whether the following series converges or diverges. If it converges, find its sum.

(1) $\sum_{n=1}^{\infty} \frac{7}{4^n}$, (2) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+2}$, (3) $\sum_{n=1}^{\infty} (-1)^n \frac{5}{3^n}$, (4) $\sum_{n=1}^{\infty} \frac{2}{3^n}$

(5) $\sum_{n=1}^{\infty} 2^n$, (6) $\sum_{n=1}^{\infty} \frac{4^{n+2}}{9^{n-1}}$, (7) $\sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^{n-1}$, (8) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n+1}}$

(9) $\sum_{n=1}^{\infty} 2^{-n}\, 3^{n-1}$, (10) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{e}}$, (11) $\sum_{n=1}^{\infty} (\sqrt{2})^{n-1}$, (12) $\sum_{n=1}^{\infty} 5^{n-1}$

(13) $\sum_{n=1}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$, (14) $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} - \frac{3}{2^n}\right)$, (15) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{5}}\right)^{n-1}$, (16) $\sum_{n=1}^{\infty} (\sqrt{5})^{n-2}$

(17) $\sum_{n=0}^{\infty} \frac{2^n - 3}{3^n}$, (18) $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$, (19) $\sum_{n=1}^{\infty} \frac{8}{3^{n-1}}$, (20) $\sum_{n=0}^{\infty} \frac{1 + 3^n}{2^n}$

(21) $\sum_{n=1}^{\infty} (-1)^{n+1}$, (22) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{5^{n-1}}$, (23) $\sum_{n=1}^{\infty} \left((-1)^{n+1} + \frac{1}{3^n}\right)$

(II) Determine whether the following series converges or diverges.

(1) $3 + \frac{3}{4} + \ldots + \frac{3}{4^{n-1}} + \ldots$. (2) $1 + \frac{1}{3} + \ldots + \frac{1}{3^{n-1}} + \ldots$.

(3) $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots$. (4) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

(5) $1 + \left(\frac{-1}{\sqrt{5}}\right) + \ldots + \left(\frac{1}{\sqrt{5}}\right)^{n-1} + \ldots$. n (6) $1 + \frac{e}{3} + \frac{e^2}{9} + \frac{e^3}{27} + \ldots$.

(7) $0.37 + 0.0037 + \ldots + \frac{37}{(100)^n} + \ldots$

(III) Determine whether the following series diverges or needs further investigation.

(1) $\sum_{n=1}^{\infty} \frac{3n}{5n - 1}$ (2) $\sum_{n=1}^{\infty} \frac{n^3}{3n^3 - 1}$ (3) $\sum_{n=1}^{\infty} \frac{1}{e^n + 1}$

(4) $\sum_{n=1}^{\infty} \frac{n^2}{n + 1}$ (5) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ (6) $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

(7) $\sum_{n=1}^{\infty} \frac{1}{2 + (0.5)^n}$ (8) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3\sqrt{n} + 5}$ (9) $\sum_{n=1}^{\infty} n^2$

(10) $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$ (11) $\sum_{n=1}^{\infty} \ln\left(\frac{2n}{7n - 8}\right)$ (12) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

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3- Positive–Term Series

In the previous section, the convergence or divergence of several series obtained by finding a formula for the *nth* partial sum and determining whether the limit of S_n as $n \to \infty$ exist or not. Unfortunately, except in special series such as a geometric series or a telescoping series, it is often impossible to find an explicit formula for S_n. However, we can develop tests for convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$ that use the *nth* term a_n.
These tests will tell us only whether the sum of the series exists or not and if will not give us this sum. However, consider only series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$, for every n (**positive-term series**).

The convergence or divergence of other series can often be determined from that of a related positive-term series.

For the positive term series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$, for every n, the sequence of partial sums

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_n = a_1 + a_2 + \ . \ . \ . + a_n$, $S_{n+1} = S_n + a_{n+1}$,
is monotonic increasing sequence

***Theorem* : 8**

> If $\sum_{n=1}^{\infty} a_n$ is a positive-term series and if there exists a number *M* such that : $S_n = a_1 + a_2 + \ . \ . \ . + a_n \leq M$ for every n , then the series converges and has a sum $S \leq M$. If no such *M* exists, the series diverges.

3-1 The Integral Test

We may use the n^{th} term a_n of a series $\sum_{n=1}^{\infty} a_n$ to define a function *f* such that $f(n) = a_n$ for every positive integer n. In some cases, If we replace *n* with *x*, we obtain a function that is defined for every real number $x \geq 1$.

The next result shows that if a function *f* obtained in this way satisfies certain conditions, then we may use the improper integral $\int_1^{\infty} f(x)\,dx$ to test the series $\sum_{n=1}^{\infty} f(n)$ for convergence or divergence.

Integral Test

If $\sum_{n=1}^{\infty} a_n$ is a series, let $f(n) = a_n$ and let $f(x)$ be the function obtained by replacing n by x. If $f(x)$ is positive-valued, continuous, and decreasing for every real number $x \geq 1$, then the series $\sum_{n=1}^{\infty} a_n$

(i) converges if: $\int_1^{\infty} f(x)\ dx$ converges. , (ii) diverges if $\int_1^{\infty} f(x)\ dx$ diverges

***Example* : 16**

Prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution

Let $f(x) = \frac{1}{x}$, then $f'(x) = \frac{-1}{x^2} < 0 \quad \forall x \geq 1$. Since $f(x)$ is positive valued, continuous, and decreasing for $x \geq 1$, we can apply the integral test

$$\int_1^{\infty} \frac{1}{x}\ dx = \lim_{t \to \infty} \ln x \Big]_1^t = \lim_{t \to \infty} (\ln t - 1) = \infty \qquad \text{(diverges).}$$

Then the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

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***Example* : 17**

Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution

Let $f(x) = \frac{1}{x^2}$, then $f'(x) = \frac{-2}{x^3} < 0 \quad \forall x \geq 1$. Since $f(x)$ is positive valued, continuous, and decreasing function for $x \geq 1$, then

$$\int_1^{\infty} \frac{1}{x^2}\ dx = \lim_{t \to \infty} \frac{-1}{x}\Big]_1^t = \lim_{t \to \infty} \left(\frac{-1}{t} + 1\right) = 1 \qquad \text{(converges).}$$

Then the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

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***Example* : 18**

Discuss the convergence of the series $\sum\limits_{n=1}^{\infty} n\ e^{-n^2}$

Solution

Let $f(x) = x\ e^{-x^2}$, $f'(x) = -(2x^2+1)e^{-x^2} < 0 \quad \forall x \geq 1.$

Since $f(x)$ is positive valued, continuous, and decreasing function for $x \geq 1$, then

$$\int_1^{\infty} x\ e^{-x^2}\ dx = \lim_{t\to\infty} \frac{-1}{2} e^{-x^2}\Big]_1^t = \frac{1}{2}\lim_{t\to\infty}(e^{-1} - e^{-t^2}) = \frac{1}{2e} < \infty \qquad \text{(converges)}.$$

Then the series $\sum\limits_{n=1}^{\infty} n\ e^{-n^2}$ converges.

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We can use the integral test to prove the following theorem which may be used as a test for convergence or divergence.

***Theorem* : 9**

The *p*-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots,$

(i) converges if $p > 1$. (ii) diverges if $p \leq 1$.

Proof

Let $f(x) = \frac{1}{x^p}$

If $p > 1$, we have : $\int_1^{\infty} x^{-p}\ dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big]_1^t = \frac{1}{p-1}$,

which is finite for $p > 1$, . Hence the p-series converges if $p > 1$, .

If $p = 1$, we have $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic divergent series.

If $0 < p < 1$, then $1 - p > 0$, and,

$$\int_1^{\infty} x^{-p}\ dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big]_1^t = \frac{1}{p-1}(\infty - 1) = \infty, \text{ (diverges)}.$$

Hence the p-series diverges if $p \leq 1$.

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Example **: 19**

Determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$, (b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$, (c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^3}}$, (d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

Solution

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$. p-series with $p = \frac{2}{3} < 1$, divergent series.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ p-series with $p = 3 > 1$, convergent series.

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^3}}$ p-series with $p = \frac{3}{5} < 1$, divergent series .

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ p-series with $p = \frac{3}{2} > 1$, convergent series.

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3-2 The Comparison Tests

The next test allows us to use convergent (divergent) series to establish the convergence (divergence) of other series.

Basic Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series,

i) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \le b_n$ for every positive integer *n*, then : $\sum_{n=1}^{\infty} a_n$ converges.

ii) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \ge b_n$ for every positive integer *n*, then: $\sum_{n=1}^{\infty} a_n$ diverges.

Example **: 20**

Determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2 + 5^n}$, (b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n - 1}}$

Solution

(a) For every $n \geq 1,\qquad \dfrac{1}{2+5^n} < \dfrac{1}{5^n} = \left(\dfrac{1}{5}\right)^n.$

The series $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{5}\right)^n$ is convergent (geometric series with $r = \dfrac{1}{5} < 1$), then the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{2+5^n}$ is convergent.

(b) For every $n \geq 2,\qquad \dfrac{1}{\sqrt{n-1}} > \dfrac{1}{\sqrt{n}}\quad .$

The series $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ is divergent (p-series with $r = \dfrac{1}{2} < 1$), then the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{\sqrt{n-1}}$ is divergent.

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Example **: 21**

Determine whether the following series converges of diverges

$$2 + \frac{2}{3} + 1 + \frac{1}{5} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{n!} + \ldots \quad .$$

Solution

By ignoring the first four terms, we have, $\dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{n!} + \ldots \quad .$

Since, $\dfrac{1}{2!} = \dfrac{1}{2} \leq \dfrac{1}{2}$, $\quad \dfrac{1}{3!} = \dfrac{1}{6} \leq \dfrac{1}{2^2}$, $\quad \dfrac{1}{4!} = \dfrac{1}{24} \leq \dfrac{1}{2^8}$, $\;\ldots$ and so on,

then the remainder of this series from the fifth term is less than the convergent geometric series $\sum\dfrac{1}{2^n}$ $\left(r = \dfrac{1}{2} < 1\right)$. Then this series is convergent.

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To apply the basic comparison test we need to have on hand a list of series that are known to converge and a list of series that are known to diverge and then prove that either $a_n \leq b_n$ or $a_n \geq b_n$. This is very difficult if a_n is a complicated expression. The following comparison test is often easier to apply, because after deciding on $\sum b_n$, we need only take of the quotient a_b / b_n as $n \to \infty$.

Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series. If there is a positive real number *c* such that : $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both series diverge.

If the limit equal $= 0$ *or* ∞, it may be possible to determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges by using the comparison test.

To find a suitable series $\sum_{n=1}^{\infty} b_n$ to use in limit comparison test when a_n is a quotient, a good procedure is to delete all terms in the numerator and the denominator of a_n except those that have the greatest effect on the magnitude.

We may also replace any constant factor *c* by 1.

Example **: 22**

Determine whether the following series converges or diverges,

(a) $\sum_{n=1}^{\infty} \frac{2n+3}{(n+2)^2}$, (b) $\sum_{n=1}^{\infty} \frac{n}{n+1}$, (c) $\sum_{n=1}^{\infty} \frac{30+2n}{n^3+10}$, (d) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Solution

(a) Let $a_n = \frac{2n+3}{n^2+4n+4}$ and $b_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{n^2+4n+4} = 2 > 0 .$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{2n+3}{(n+2)^2}$ diverges.

(b) Let $a_n = \frac{n}{n+1}$ and $b_n = 1$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \succ 0 .$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1$ diverges by the *nth*-term test,, then $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

(c) Let $a_n = \dfrac{30 + 2n}{n^3 + 10}$ and $b_n = \dfrac{1}{n^2}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{30n^2 + 2n^3}{n^3 + 10} = 2 \succ 0 .$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent *p*-series, then $\sum_{n=1}^{\infty} \dfrac{30 + 2n}{n^3 + 10}$ converges.

(d) Let $a_n = \dfrac{1}{2^n - 1}$ and $b_n = \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - (1/2)^n} = 1 \succ 0 .$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\dfrac{1}{2}\right)^n$ is a geometric convergent series, then $\sum_{n=1}^{\infty} \dfrac{1}{2^n - 1}$ converges.

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Example **: 23**

Determine whether the following series converges or diverges,

(a) $\sum_{n=1}^{\infty} \dfrac{3}{\sqrt{n^2 + n + 1}}$, (b) $\sum_{n=1}^{\infty} \dfrac{\sqrt[3]{n^2 + 4}}{2n^2 - n - 1}$

Solution

(a) Let $a_n = \dfrac{3}{\sqrt{n^2 + n + 1}}$ and $b_n = \dfrac{1}{n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n}{\sqrt{n^2 + n + 1}} = 3 > 0 .$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \dfrac{3}{\sqrt{n^2 + n + 1}}$ diverges.

(b) Let $a_n = \dfrac{\sqrt[3]{n^2 + 4}}{2n^2 - n - 1}$ and $b_n = \dfrac{n^{2/3}}{n^2} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n^6 + 4n^2}}{2n^2 - n - 1} = \frac{1}{2} > 0 .$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$ is a convergent *p*-series

Then $\sum_{n=1}^{\infty} \dfrac{\sqrt[3]{n^2 + 4}}{2n^2 - n - 1}$ converges.

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***Example* : 24**

Does the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges ?

Solution

We know that $\ln n$ increases more slowly than n as $n \to \infty$, i.e.

$\ln n < n$ as $n \to \infty$. Let $a_n = \frac{\ln n}{n^3}$ and $b_n = \frac{n}{n^3} = \frac{1}{n^2}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \quad \text{(teast fails)}$$

Then we can not decide the convergence by this test, we can use the basic comparison test as follows , $\frac{\ln n}{n^3} \le \frac{n}{n^3} = \frac{1}{n^2}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent *p*-series

Then $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ is a convergent series.

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3-3 The Ratio And Root Tests

As we said before, it is not always possible to discuss the convergence of the series by using the basic comparison and limit comparison tests for some complicated expressions. For the integral test to be applied, the terms of the series must be decreasing as n increasing, or we might not find a formula for the *nth* term test. These conditions often rule out series that involve factorials and other complicated expressions. The following two tests can be used to determine convergence or divergence when other tests are not applicable.

The first test is the ratio test which is often effective when terms of the series contain factorials or terms contains powers of *n*.

The Ratio Test

Let $\sum a_n$ be a positive-term series, and suppose that: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Then

(i) If $L < 1$, the series is convergent.

(ii) If $L > 1$ or ∞, the series is divergent.

(iii) If $L = 1$, the series may be convergent or divergent, (test fails).

***Example* : 25**

Test the following series for convergence or divergence,

(a) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$, (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

Solution

(a) $a_n = \frac{3^n}{n!}$, $a_{n+1} = \frac{3^{n+1}}{(n+1)!}$, then $\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1}$.

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$. Then the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ is convergent.

(b) $a_n = \frac{2^n}{n^2}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)^2}$, then $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = \frac{2n^2}{n^2 + 2n + 1}$.

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n^2}{n^2 + 2n + 1} = 2 > 1$.

Then the series $\sum_{n=1}^{\infty} \frac{2^2}{n^2}$ is divergent.

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***Example* : 26**

Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

$a_n = \frac{n^n}{n!}$, $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$, then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^{n+1}}{(n+1)\, n!} \cdot \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1 .$$

Then the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.

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***Example* : 27**

Determine whether the following series converges or diverges,

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{n!\,n!}{(2n)!}\quad,\quad \text{(b)}\ \sum_{n=1}^{\infty}\frac{4^n\ n!\,n!}{(2n)!}\quad,\quad \text{(c)}\ \sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$$

Solution

$$\text{(a)}\ \frac{a_{n+1}}{a_n}=\frac{(n+1)!\ (n+1)!}{(2n+2)\,!}\cdot\frac{(2n)\,!}{n!\,n!}=\frac{(n+1)\ (n+1)}{(2n+2)\ (2n+1)}=\frac{n+1}{4n+2}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n+1}{4n+2}=\frac{1}{4}<1\,.$$

Then the series $\sum_{n=1}^{\infty}\frac{n!\,n!}{(2n)!}$ is convergent.

$$\text{(b)}\ \frac{a_{n+1}}{a_n}=\frac{4^{n+1}(n+1)!\ (n+1)!}{(2n+2)\,!}\cdot\frac{(2n)\,!}{4^{\,n}\ n!\,n!}=\frac{(n+1)\ (n+1)}{(2n+2)\ (2n+1)}=\frac{2(n+1)}{2n+1}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2(n+1)}{2n+1}=1\,.\qquad\text{(test fails)}$$

So we try to discuss the convergence of this series by any other method. We note that,

$$\sum_{n=1}^{\infty}\frac{4\ n!\,n!}{(2n)!}=2+\frac{8}{3}+\frac{16}{5}+\frac{128}{35}+\ .\ .\ .$$

The elements of the sequence of partial sums are:

$$S_1=2,\ \ S_2=\frac{14}{3},\ S_3=\frac{118}{5}\ ,\ .\ .\ .\ .$$

This means that the sequence of partial sums are always grow and the series

$\sum_{n=1}^{\infty}\frac{4\ n!\,n!}{(2n)!}$ is divergent.

$$\text{(c)}\ \frac{a_{n+1}}{a_n}=\frac{2^{n+1}+5}{3^{n+1}}\cdot\frac{3^n}{2^n+5}=\frac{1}{3}\cdot\frac{2^{n+1}+5}{2^n+5}\ .$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{3}\cdot\frac{2^{n+1}+5}{2^n+5}=\frac{1}{3}\lim_{n\to\infty}\frac{2+5/2^n}{1+5/2^n}=\frac{2}{3}<1\,.$$

Then the series $\sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$ is convergent.

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Example **: 28**

For what positive values of *x* does the series converges?

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots + \frac{x^{2n-1}}{2n-1} + \ldots \quad .$$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} = \frac{2n-1}{2n+1} x^2 .$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n-1}{2n+1} x^2 = x^2 .$$

The series converges if $x^2 > 1$, then the series converges if x is positive and less than 1 and diverges if x greater than one.

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The second test is the root test which is often effective when the terms of the series contains powers of *n*.

The Root Test

Let $\sum a_n$ be a positive-term series, and suppose that: $\lim_{n\to\infty} \sqrt[n]{a_n} = L$. Then

(i) If $L < 1$, the series is convergent.

(ii) If $L > 1$ or ∞, the series is divergent.

(iii) If $L = 1$, the series may be convergent or divergent, (test fails).

Example **: 29**

Discuss the convergence of the following series

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$, (b) $\sum_{n=1}^{\infty} \frac{3^n}{n^4}$

Solution

(a) $a_n = \frac{n^2}{2^n}$, $\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{n^{2/n}}{2} = \frac{1}{2} < 1$, The series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is convergent.

(b) $a_n = \frac{3^n}{n^4}$, $\lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^4}} = \lim_{n\to\infty} \frac{3}{n^{4/n}} = 3 \succ 1$, The series $\sum_{n=1}^{\infty} \frac{3^n}{n^4}$ is divergent.

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Example **: 30**

Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^{2n}}$

Solution

$$a_n = \frac{2^{3n+1}}{n^{2n}} , \quad \lim_{n \to \infty} \sqrt[n]{\frac{2^{3n+1}}{n^{2n}}} = \lim_{n \to \infty} \frac{2^{3+\frac{1}{n}}}{n^2} = 0 < 1 .$$

Then the series $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^{2n}}$ is convergent.

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Example **: 31**

Discuss the convergence of the series $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$.

Solution

$$a_n = \left(1 - \frac{3}{n}\right)^n , \quad \lim_{n \to \infty} \sqrt[n]{\left(1 - \frac{3}{n}\right)^n} = \lim_{n \to \infty} \left(1 - \frac{3}{n}\right) = 1 \quad (\text{test fails}) .$$

The series need further investigation. Use the nth term test,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 - \frac{3}{n}\right)^n = e^{-3} \neq 0 . \text{ So the series diverges.}$$

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<u>Exercise</u> (4– 3)

(I) Use the integral test to determine whether the following series converges or diverges,

(1) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ (2) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$ (3) $\sum_{n=1}^{\infty} \frac{1}{3n + 2}$

(4) $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ (5) $\sum_{n=1}^{\infty} \frac{1}{n(2n - 5)}$ (6) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n + 5}}$

(7) $\sum_{n=1}^{\infty} \frac{1}{1 + 16n^2}$ (8) $\sum_{n=1}^{\infty} \frac{1}{n \ln(n + 1)}$ (9) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$

(10) $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 3}$ (11) $\sum_{n=0}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$. (12) $\sum_{n=1}^{\infty} \frac{3}{n^2 + 1}$

(II) Use the p-series test to determine whether the following series converges or diverges.

(1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ (2) $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ (3) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(4) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[9]{n^{11}}}$ (5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^3}}$ (6) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

(III) Use the basic comparison test to determine whether the following series converges or diverges,

(1) $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2 + 1}$ (2) $\sum_{n=1}^{\infty} \frac{8n^2 - 7}{e^n (n + 1)^2}$ (3) $\sum_{n=1}^{\infty} \frac{2 + \cos n}{n^2}$

(4) $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$ (5) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ (6) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^3 - 8n}}$

(IV) Use the limit comparison test to determine whether the following series converges or diverges,

(1) $\sum_{n=1}^{\infty} \frac{8n^2 - 7}{e^n (n + 1)}$ (2) $\sum_{n=1}^{\infty} \frac{n^5}{n^3 + 5}$ (3) $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

(4) $\sum_{n=1}^{\infty} \sqrt{\frac{n^2 + 1}{n^3 - 1}}$ (5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^3 - 8n}}$ (6) $\sum_{n=1}^{\infty} \frac{\sqrt[5]{x^2 + n - 1}}{\sqrt{x^3 + 1}}$

(V) Use the ratio test or the root test. Discuss the convergence of the following series

(1) $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ (2) $\sum_{n=1}^{\infty} \frac{2^n}{n^2+4}$ (3) $\sum_{n=1}^{\infty} \frac{4^n}{n^2}$

(4) $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ (5) $\sum_{n=1}^{\infty} \frac{1}{4n}$ (6) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

(7) $\sum_{n=1}^{\infty} \left(\frac{3n+5}{2n-7}\right)^n$ (8) $\sum_{n=1}^{\infty} \frac{n!}{(n+1)^3}$ (9) $\sum_{n=1}^{\infty} \left(1-\frac{2}{n}\right)^{n^2}$

(10) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ (11) $\sum_{n=1}^{\infty} \frac{n!}{n^3}$ (12) $\sum_{n=1}^{\infty} \frac{n^{10}+10}{n!}$

(13) $\sum_{n=1}^{\infty} \frac{n}{3^n}$ (14) $\sum_{n=1}^{\infty} \frac{5^{n+1}}{(\ln n)^n}$ (15) $\sum_{n=1}^{\infty} \left(1-\frac{2}{n}\right)^n$

(VI) Determine whether the following series converges or diverges,

(1) $\sum \frac{1+n^3}{1+5^n}$ (2) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$ (3) $\sum \frac{3n}{2n^2-7}$

(4) $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ (5) $\sum_{n=1}^{\infty} n\, e^{-n}$ (6) $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

(7) $\sum \frac{\ln n}{n^2}$ (8) $\sum_{n=1}^{\infty} \frac{1}{(\ln n)^2}$ (9) $\sum \frac{1}{n(n+1)}$

(10) $\sum_{n=1}^{\infty} \frac{n+\ln n}{n^2+1}$ (11) $\sum_{n=1}^{\infty} \frac{n^2+2^n}{2+3^n}$ (12) $\sum_{n=1}^{\infty} \frac{1-n}{n-n^2}$

(13) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n+1}}$ (14) $\sum_{n=1}^{\infty} \frac{1+3^n}{1+5^n}$ (15) $\sum_{n=1}^{\infty} \left(1-\frac{3}{n}\right)^n$

(16) $\sum_{n=1}^{\infty} \frac{4n^2-n+1}{n^3-n+3}$ (17) $\sum \frac{n}{n+1}$ (18) $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

(19) $\sum_{n=1}^{\infty} \sqrt{n+1}-\sqrt{n}$ (20) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (21) $\sum_{n=1}^{\infty} \frac{n+\sqrt{n}}{n^3+3n}$

(22) $\sum_{n=1}^{\infty} \frac{\ln n}{2n^3+1}$ (23) $\sum_{n=1}^{\infty} n^4\, e^{-n^2}$ (24) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

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4- Alternating Series, Absolute And Conditional Convergence

4-1 Alternating Series

The tests for convergence that we have discussed in the previous section can be applied only to positive-term series. We now consider infinite series that contain both positive and negative terms. One of the most important type is an alternating series, in which the terms are alternately positive and negative,

$$\sum_{n=1}^{\infty}(-1)^n\ a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^n\ a_n + \ldots \ \text{, with } a_k > 0 \text{ for every } k.$$

Alternating Series Test

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\ a_n$ is convergent if the following two conditions are satisfied;

(i) $\{a_n\}$ is decreasing. , (ii) $\lim_{n\to\infty} a_n = 0$.

There are two methods to prove (i)

(1) Directly, by proving that : $a_n - a_{n+1} \geq 0$

(2) Express a_n by $f(n)$ and replace n by x and then prove that $f(x)$ is decreasing i.e. $f'(x) \prec 0$ for every $x \geq 1$.

***Example* : 32**

Discuss the convergence of the following series,

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\ \frac{1}{n}$, (b) $\sum_{n=1}^{\infty}(-1)^{n-1}\ \frac{1}{n^2}$

Solution

(a) $a_n = \frac{1}{n}$, $f(n) = \frac{1}{n}$, $f(x) = \frac{1}{x}$

(i) $f'(x) = \frac{-1}{x^2} \prec 0$ for all $x \geq 1$, then $\{a_n\}$ is decreasing

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Then the series $\sum_{n=1}^{\infty}(-1)^n\ \frac{1}{n}$ is convergent.

(b) $a_n = \frac{1}{n^2}$, $f(n) = \frac{1}{n^2}$, $f(x) = \frac{1}{x^2}$

(i) $f'(x) = \frac{-2}{x^3} \prec 0$ for all $x \geq 1$, then $\{a_n\}$ is decreasing

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0.$

Then the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ is convergent

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Example **: 33**

Discuss the convergence of the following series,

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n^2 - 3}$, (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n - 3}$

Solution

(a) $a_n = \frac{2n}{4n^2 - 3}$, $f(n) = \frac{2n}{4n^2 - 3}$, $f(x) = \frac{2x}{4x^2 - 3}$

(i) $f'(x) = \frac{(4x^2 - 3)(2) - (2x)(8x)}{(4x^2 - 3)^2}$

$= -\frac{8x^2 + 6}{(4x^2 - 3)^2} < 0$ for all $x \geq 1$, then $\{a_n\}$ is decreasing

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n^2 - 3} = 0.$

Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n^2 - 3}$ is convergent.

(b) $a_n = \frac{2n}{4n - 3}$, $f(n) = \frac{2n}{4n - 3}$, $f(x) = \frac{2x}{4x - 3}$

(i) $f'(x) = \frac{(4x - 3)(2) - (2x)(4)}{(4x - 3)^2}$

$= -\frac{6}{(4x - 3)^2} < 0$ for all $x \geq 1$, then $\{a_n\}$ is decreasing

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n - 3} = \frac{1}{2} \neq 0.$

Then the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n - 3}$ is divergent.

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4-2 Absolute And Conditional Convergence

Note that: in example (1) we obtain,

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges while $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges while $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges too.

Now we ask our-self what is the difference between theses two series? , the answer of this question leads us to the following definitions.

Definition **: 6**

(1) A series $\sum_{n=1}^{\infty} a_n$ is **absolutely** convergent if the series,

$$\sum_{n=1}^{\infty}|a_n| = |a_1| + |a_2| + \ldots + |a_n| + \ldots \quad \text{is convergent.}$$

(2) A series $\sum_{n=1}^{\infty} a_n$ is **conditionally** convergent if $\sum_{n=1}^{\infty} a_n$ is convergent and

$\sum_{n=1}^{\infty}|a_n|$ is divergent.

According to these definitions, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ is absolutely convergent while the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is conditionally convergent.

The following theorem tells us that absolute convergence implies convergence of the series.

Theorem **: 10**

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

Corollary **: 2**

If $\sum_{n=1}^{\infty} a_n$ diverges, the $\sum_{n=1}^{\infty}|a_n|$ diverges.

<u>Note that:</u> If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ may be converges or diverges.

***<u>Example</u>* : 34**

Determine whether the following series converges or diverges,

$$\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7} - \frac{1}{2^8} + \dots \; .$$

<u>Solution</u>

The series is neither alternating nor geometric no positive-term, so none of the earlier tests can be applied. Let us consider the series of absolute values,

$\sum |a_n| = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \dots$ .which is a geometric series with $r = 1/2 \prec 1$, thus the given series is absolutely convergent and hence the given series is convergent.

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***<u>Example</u>* : 35**

Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

<u>Solution</u>

The series is neither alternating nor geometric nor positive term, so

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} .$$

Since $|\sin n| \le 1$, then the series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and by the basic comparison test, then

the series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converges, i.e. the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent, and hence, the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

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Example **: 36**

Discuss the convergence of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{3n+2}$

Solution

$$a_n = \frac{n}{3n+2}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{3n+2} = \frac{1}{3} \neq 0.$$

Then the alternating series, $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{3n+2}$ is divergent.

By the n^{th} term test we see that, $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\dfrac{n}{3n+2}\right| = \sum_{n=1}^{\infty}\dfrac{n}{3n+2}$ is divergent.

We see from the preceding discussion that an alternating series may be classified in exactly one of the following ways :

** absolutely convergent series

** conditionally convergent series

** divergent series

The following two tests may be used to investigate absolute convergence.

Ratio Test For Absolute Convergence

Let $\sum a_n$ be a series of non zero terms, and suppose that: $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L$., Then

(i) If $L \prec 1$, the series is absolutely convergent.

(ii) If $L \succ 1$ or ∞, the series is divergent.

(iii) If $L = 1$, the series may be absolutely convergent, conditionally convergent or divergent, (test fails).

Root Test For Absolute Convergence

Let $\sum a_n$ be a series of non zero terms, and suppose: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$, Then

(i) If $L \prec 1$, the series is absolutely convergent.

(ii) If $L \succ 1$ or ∞, the series is divergent.

(iii) If $L = 1$, the series may be absolutely convergent, conditionally convergent or divergent, (test fails).

Example **: 37**

Determine whether the following series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n^2+4}{2^n}$, (b) $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n+1}$

(c) $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n+1}{n^2}$, (d) $\sum_{n=1}^{\infty}(-1)^{n-1}(0.3)^n$

Solution

(a) $|a_n| = \dfrac{n^2+4}{2^n}$, $|a_{n+1}| = \dfrac{(n+1)^2+4}{2^{n+1}}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2+4}{2^{n+1}} \cdot \frac{2^n}{n^2+4}\right| = \frac{1}{2}\left|\frac{n^2+2n+5}{n^2+4}\right| .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}\lim_{n\to\infty}\left(\frac{n^2+2n+5}{n^2+4}\right) = \frac{1}{2}\ (1) = \frac{1}{2} < 1 .$$

Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n^2+4}{2^n}$ is absolutely convergent.

(b) $|a_n| = \dfrac{n}{n+1}$, $|a_{n+1}| = \dfrac{n+1}{n+2}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{n+2} \cdot \frac{n+1}{n}\right| = \left|\frac{n^2+2n+1}{n^2+2n}\right| .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n^2+2n+1}{n^2+2n}\right) = 1 . \quad (\textit{test fails})$$

The series needs more investigations, by the second condition for alternating series test,

$$a_n = \frac{n}{n+1}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0.$$

The series is divergent.

(c) $|a_n| = \dfrac{1+n}{n^2}$, $|a_{n+1}| = \dfrac{2+n}{(n+1)^2}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+2}{(n+1)^2} \cdot \frac{n^2}{n+1}\right| = \left|\frac{n^3+2n^2}{(n+1)^3}\right| .$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n^3+2n^2}{(n+1)^3}\right) = 1 . \quad (\textit{test fails}) \cdot$$

By alternating series test,

(i) $f(x) = \frac{1 + x}{x^2}$, $f'(x) = \frac{-x - 2}{x^3} < 0$ for all $x \geq 1$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1 + n}{n^2} = 0.$

Then $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n + 1}{n^2}$ converges, but the absolute value of the series

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n + 1}{n^2}\right| = \sum_{n=1}^{\infty} \left|\frac{n + 1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a sum of convergent and divergent series, i.e. which is divergent series. Then

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n + 1}{n^2}$ is conditionally convergent series.

(d) $|a_n| = (0.3)^n$, $|a_{n+1}| = (0.3)^{n+1}$ then .

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{(0.3)^{n+1}}{(0.3)^n}\right) = 0.3 < 1 .$$

Then the series $\sum_{n=1}^{\infty}(-1)^{n-1} (0.3)^n$ is absolutely convergent.

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Exercise (4– 4)

(I) Determine whether the following series converges or diverges,

(1) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{1/3}}$, (2) $\sum_{n=1}^{\infty}(-1)^{n+1}\, n\, 5^{-n}$, (3) $\sum_{n=1}^{\infty}(-1)^{n+1}$

(4) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n^3+17}$, (5) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2+7}$, (6) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n^3}$

(7) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(1+e^{-n}\right)$, (8) $\sum_{n=1}^{\infty}(-1)^{n+1}\, n\sin(1/n)$, (9) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{3^n}$

(10) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$, (11) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{2n}+1}{e^{2n}-1}$, (12) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^n-1}{e^n+1}$

(13) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$, (14) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+1}{7^n}$,(15) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

(16) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$, (17) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{\ln n}$, (18) $\sum_{n=1}^{\infty}(-1)^{n+1}(3)$

(19) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\tan^{-1} n}{n^2}$, (20) $\sum_{n=1}^{\infty}(-1)^{n+1}\, n\sin(1/n)$

(II) Determine whether the following series is absolutely converges, conditionally convergent, or divergent,

(1) $\sum_{n=1}^{n}\frac{2-n}{n^3}$, (2) $\sum_{n=1}^{n}(-1)^{n+1}\frac{7}{n!}$, (3) $\sum_{n=1}^{n}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)$

(4) $\sum_{n=1}^{n}(-1)^{n+1}5^n$, (5) $\sum_{n=1}^{n}\left(\frac{1}{2^n}-1\right)$, (6) $\sum_{n=1}^{n}(-1)^{n+1}\frac{\sqrt{n}}{n+1}$

(7) $\sum_{n=1}^{n}\left(\frac{-n}{2^n}\right)$, (8) $\sum_{n=1}^{n}\frac{e^{3n}}{n^{3n}}$, (9) $\sum_{n=1}^{n}(-1)^{n-1}\frac{1}{(2n)!}$

(10) $\sum_{n=1}^{n}(-1)^{n+1}\frac{n}{n^3+1}$, (11) $\sum_{n=1}^{n}(5)^{-n}$, (12) $\sum_{n=1}^{n}(-1)^{n+1}\frac{3^n\, n^2}{n!}$

(13) $\sum_{n=1}^{n}(-1)^{n-1}\frac{\sqrt[3]{n}}{n+1}$, (14) $\sum_{n=1}^{n}(-1)^{n+1}\frac{n^{10}}{(2n)!}$, (15) $\sum_{n=1}^{n}(-1)^{n-1}\frac{(n+1)^2}{n^5+1}$

(16) $\sum_{n=1}^{n}(-1)^{n+1}\frac{1}{n\sqrt{n}}$, (17) $\sum_{n=1}^{n}(-1)^{n-1}\frac{\sqrt{2}}{n!}$, (18) $\sum_{n=1}^{n}\frac{\sin n}{n^2}$

(19) $\sum_{n=1}^{n}(-1)^{n-1}\frac{1+4^n}{1+3^n}$, (20) $\sum_{n=1}^{n}\frac{(n^2+1)^n}{(-n)^n}$, (21) $\sum_{n=1}^{n}(-1)^{n-1}\frac{n^n}{e^n}$

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5- <u>Power Series</u>

The power series in (*x* - *c*) is a series of the form,

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n (x-c)^n + \ldots ,$$

where c is called the center of the series and $a_0, a_1, \ldots , a_n, \ldots$ are the coefficients of the series. When the center c = 0, the power series reduces to,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots ,$$

In a special case when the coefficients $a_n = 1$, for all *n*, the series takes the form,

$$\sum_{n=0}^{\infty} (x-c)^n = 1 + (x-c) + (x-c)^2 + \ldots + (x-c)^n + \ldots ,$$

which is a geometric series. This series converges if $|x - c| < 1$, which gives

$$-1 < (x - c) < 1, \quad i.e. \quad c - 1 < x < c + 1$$

and it converges to : $\dfrac{1}{1-(x - c)}$.

The main objective of this section is to determine all values of *x* for which the power series converges. Every power series in (*x* - *c*) converges if $x = c$, since

$$a_0 + a_1 (0) + a_2 (0)^2 + \ldots + a_n (0)^n + \ldots = a_0 .$$

To find other values of *x* that produce convergent series, we often use the ratio test for absolute convergence.

***<u>Example</u>* : 38**

Find all values of *x* for which the following power series is absolutely convergent:

$$\sum_{n=1}^{\infty} \frac{(x - 3)^n}{n}$$

<u>Solution</u>

If we let $u_n = \dfrac{(x - 3)^n}{n}$, then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left(\frac{(x - 3)^{n+1}}{n+1} \frac{n}{(x - 3)^n}\right).$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right) |x-3| = |x-3| .$$

For convergence $|x - 3| < 1$,

i.e. $-1 < x - 3 < 1$, i.e. $2 < x < 4$.

The series is divergent if : $|x - 3| > 1$, *i.e.* if $x < 2$ or $x > 4$.

If $|x - 3| = 1$, the series may be converges or diverges, so we must discuss the convergence at $x = 2$ and at $x = 4$.

At $x = 2$: $\sum\limits_{n=1}^{\infty} \frac{(x - 3)^n}{n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is a convergent alternating series.

At $x = 4$: $\sum\limits_{n=1}^{\infty} \frac{(x - 3)^n}{n} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ which is a divergent (harmonic) series.

So, the power series is absolutely convergent for every x in the semi-open interval $[2, 4)$ and diverges everywhere.

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Example **: 39**

Find all values of *x* for which the following series is absolutely convergent. $\sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$

Solution

If we let $u_n = \frac{x^n}{n!}$, then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left(\frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n}\right) = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 .$$

The limit is less than 1 for every value of *x*, and hence, the power series is absolutely convergent for every real number *x*.

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Example **: 40**

Find all value of *x* for which the following series is convergent $\sum\limits_{n=1}^{\infty} n!\ x^n$

Solution

If we let $u_n = n!\ x^n$, then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left(\frac{(n+1)!\ x^{n+1}}{n!\ x^n}\right) = \lim_{n\to\infty} |(n+1)\, x| = \infty \text{ for all values of } x$$

except at $x = 0$. Hence the power series is convergent only if $x = 0$.

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The following theorem will describe the solutions of the above examples in more general way.

***Theorem* : 11**

If $\sum_{n=0}^{\infty} a_n (x-c)^n$ is a power series, then exactly one of the following is true;

(i) The series converges only if $x - c = 0$, i.e., if $x = c$.

(ii) The series is absolutely convergent for every x.

(iii) There is a number $r > 0$ such that the series is absolutely convergent if x is in the open interval $(c - r, c + r)$ and divergent if: $x < c - r$ or $x > c + r$

In case (iii) of the above theorem, the endpoints $c - r$ and $c + r$ of the interval must be investigated separately.

***Definition* : 7**

The number r in theorem (11) is called the radius of convergence of the series. The totality of numbers for which a power series converges is called its interval of convergence. If the radius of convergence r is positive, then the interval of convergence is one of the following $(c-r, c+r)$, $(c-r, c+r]$, $[c-r, c+r)$, $[c-r, c+r]$.

In example (38) above, the radius of convergence is 1 and the interval of convergence is $[2, 4]$. In example (39), the interval of convergence is $(-\infty, \infty)$ and we write $r = \infty$. In example (40), $r = 0$.

Example **: 41**

Find the radius and interval of convergence of the power series, $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

Solution

Let $u_n = \frac{x^n}{\sqrt{n}}$, then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left(\frac{x^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{x^n}\right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} |x| = |x|$$

The series converges for $|x| < 1$ *i.e.* $-1 < x < 1$ and diverges for $|x| > 1$ *i.e.* $x < -1$ or $x > 1$.

At $x = 1$: $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which is a divergent *p*-series $\left(p = \frac{1}{2} < 1\right)$.

At $x = -1$: $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ which is a convergent alternating series.

Then the radius of convergence $r = 1$ and the interval of convergence is $[-1, 1)$.

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Exercise (4– 5)

(I) Find the radius and interval of convergence for the following series

(1) $\sum_{n=0}^{\infty} \frac{n^2 x^n}{3^n}$, (2) $\sum_{n=0}^{\infty} \frac{1}{n+4} x^n$, (3) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{\sqrt{n}}$

(4) $\sum_{n=0}^{\infty} \frac{1}{n^3+1} x^n$, (5) $\sum_{n=0}^{\infty} \frac{10^n x^n}{n!}$, (6) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n} x^{n+1}$

(7) $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{n^2 (x-3)^n}{n+1}$, (8) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} x^n$, (9) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(-4)^n}$

(10) $\sum_{n=1}^{\infty} \frac{1}{4^n} \frac{x^n}{\sqrt{n}}$, (11) $\sum_{n=0}^{\infty} \frac{(3n)!}{(2n)!} x^n$, (12) $\sum_{n=0}^{\infty} \frac{10^{n+1} x^n}{3^{2n}}$

(13) $\sum_{n=0}^{\infty} \frac{(3x+4)^n}{\sqrt{3n+4}}$, (14) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n(n+1)}$, (15) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{25\, n}$

(16) $\sum_{n=0}^{\infty} \frac{(x+3)^n}{2n+1}$, (17) $\sum_{n=0}^{\infty} \frac{n!}{100^n} x^n$, (18) $\sum_{n=0}^{\infty} \frac{3^{2n} (x-2)^n}{n+1}$

(19) $\sum_{n=0}^{\infty} \frac{n (x-1)^{2n}}{3^{2n-1}}$, (20) $\sum_{n=1}^{\infty} \frac{\ln n \, (x-e)^n}{e^n}$, (21) $\sum_{n=0}^{\infty} \frac{10^{n+1} x^n}{4^{2n}}$

(22) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\, 5^n}$, (23) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$, (24) $\sum_{n=1}^{\infty} (-1)^n \frac{e^{n+1} (x-1)^n}{n^n}$

(II) Find the radius of convergence of the following power series for positive integers

(1) $\sum_{n=0}^{\infty} \frac{(n+1)!\, (x-6)^n}{10^n}$, (2) $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$

(III) Find the radius of convergence of the following power series for positive integers c and d.

(1) $\sum_{n=0}^{\infty} \frac{(n+c)!\, x^n}{n!\, (n+d)!}$, (2) $\sum_{n=0}^{\infty} \frac{(cn)!\, x^n}{(n!)^c}$

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6- Taylor And Maclaurin Series

A power series $\sum a_n (x-c)^n$ or $\sum a_n x^n$ determines a function $f(x)$ whose domain is the interval of convergence of the series. Specifically, for each *x* in this interval, we let $f(x)$ equal the sum of the series and we say that $\sum a_n (x-c)^n$ or $\sum a_n x^n$ is a power series representation for $f(x)$.

Numerical computations using power series provide the basis for the design of calculators and construction of mathematical tables. In addition to this use, differentiation and integration can be performed by using the power series representation.

One of the most important power series representation for a function $f(x)$ is the Taylor series.

Taylor Series

If a function $f(x)$ ha a power series representation, $f(x) = \sum a_n (x-c)^n$ with radius of convergence $r > 0$, then $f^{(k)}(c)$ exists for every positive integer *k* and $a_n = f^{(n)}(c)/n!$.

Thus

$$f(x) = f(c) + f'(c).(x-c) + \frac{f''(c)}{2!}.(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}.(x-c)^n + \ldots$$

A special case from Taylor series if at $c = 0$ is the Maclaurin series.

Maclaurin Series

If a function $f(x)$ ha a power series representation, $f(x) = \sum a_n x^n$ with radius of convergence $r \succ 0$, then $f^{(k)}(0)$ exists for every positive integer *k* and $a_n = f^{(n)}(0)/n!$.

Thus

$$f(x) = f(0) + f'(0).x + \frac{f''(0)}{2!}.x^2 + \ldots + \frac{f''(0)}{2!}.x^2 + \ldots$$

***Example* : 42**

Find the Taylor series for the function $f(x)= \sin x$ in a power series at $x=\pi/6$.

Solution

$$f(x) = \sin x, \qquad f\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad a_0 = \frac{1}{2}$$

$$f'(x) = \cos x, \qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad a_1 = \frac{\sqrt{3}}{2}$$

$$f''(x) = -\sin x, \qquad f''\left(\frac{\pi}{6}\right) = \frac{-1}{2} \qquad a_2 = \frac{-1}{2(2!)}$$

$$f'''(x) = -\cos x, \qquad f'''\left(\frac{\pi}{6}\right) = \frac{-\sqrt{3}}{2} \qquad a_3 = \frac{-\sqrt{3}}{2(3!)}$$

Then : $$\sin x=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{2}\frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{2}\frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3+\dots .$$

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***Example* : 43**

Find the Maclaurin series representation for the function $f(x)= e^x$

Solution

$$f(x) = f'(x) = f''(x) = \dots = f^{(n)}(x) = e^x$$

Thus, $f(0) = f'(0) = f''(0) = \dots = f^{(n)}(0) = 1$

Then : $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots .$$

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***Example* : 44**

Find the Maclaurin series for the function $f(x)= \sin x$

Solution

From example (42), we obtain,

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 1$

Then :

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots .$$

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Now the question that arises here is, what conditions on a function guarantee that a power series representation exists ? We shall next obtain such conditions. Let us begin with the following definition.

<u>Definition</u> **: 8**

Let c be a real number and let f be a function that has n derivatives at c. The *nth*-degree Taylor polynomial $P_n(x)$ of f at c is,

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n .$$

and the *nth* degree Maclaurin polynomial of f at 0 is,

$$P_n(x) = f(0) + f'(0).x + \frac{f''(0)}{2!}.x^2 + \ldots + \frac{f^{(n)}(0)}{n!}.x^n .$$

Note that $P_n(x)$ is the $(n+1)^{st}$ partial sum of the series

(c−r, c+r); [c−r, c+r]; [c−r, c+r); (c−r, c+r]

<u>Theorem</u> **: 12**

Let f have n+1 derivative throughout an interval containing *c*. If *x* is any number in the interval that is different from *c*, then there is a number *z* between *c* and *x* such that: $f(x) = P_n(x) + R_n(x)$,where, $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}$.

The term R_n in theorem (12) is called the Taylor remainder of f at c. If $c = 0$ is $R_n(x)$ the Maclaurin remainder of f.

Now, the sufficient conditions for the existence of power series representation for a function are given by the following theorem.

<u>Theorem</u> **: 13**

Let f(x) have derivatives of all orders throughout an interval containing *c*, and let $R_n(x)$ be the Taylor remainder of f at *c*. If $\lim_{n\to\infty} R_n(x) = 0$ for every *x* in the interval, then f(x) is represented by the Taylor series for f(x) at *c*.

<u>Example</u> **: 45**

Show that the Maclaurin series obtained in example (44) represents $\sin x$ for every real number *x*.

<u>Solution</u>

In example (3), If *n* is a positive integer, then either

$$\left|f^{(n+1)}(x)\right| = |\cos x| \qquad \text{or} \qquad \left|f^{(n+1)}(x)\right| = |\sin x|$$

Hence $\left|f^{(n+1)}(z)\right| \le 1$ for every number *z* and

$$|R_n(x)| = \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!} = |x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}$$

$\lim\limits_{x\to\infty} |R_n(x)| = 0$, consequently $\lim\limits_{x\to\infty} R_n(x) = 0$, and the Maclaurin series in example (45) represents $\sin x$ for every number *x*.

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<u>Example</u> **: 46**

Show that the Maclaurin series obtained in example (2) represents e^x for every number *x*.

<u>Solution</u>

For $f(x) = e^x$, $f^{(n+1)}(z) = e^z$,

We obtain, $$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} = \frac{e^z}{(n+1)!}x^{n+1}$$

where *z* in a number between 0 and *x*. If $0 < x$, then $e^z \prec e^x$ since the natural exponential function is increasing, and hence for every positive integer *n*,

$$0 < R_n(x) < \frac{e^z}{(n+1)!}x^{n+1}$$

$$\lim_{n\to\infty} \frac{e^z}{(n+1)!}x^{n+1} = e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0,$$

and by Sandwich theorem Limit $\lim\limits_{x\to\infty} R_n(x) = 0$,

If $x < 0$, then $z < 0$ and hence $e^z < e^0 = 1$.. Consequently

$$0 < |R_n(x)| < \left|\frac{x^{n+1}}{(n+1)!}\right|$$

and hence $R_n(x)$ has the limit 0 as $n \to \infty$. It follows that the power series representation for e^x is valid for all non-zero *x*. Finally, if $x = 0$, then the series reduces to $e = 1$.

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Example **: 47**

Show that the function $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

does not have the Maclaurin series representation.

Solution

It is easy to prove that $f'(0) = f''(0) = f'''(0) = ... = f^{(n)}(0) = 0$ for every positive integer *n*. If *f(x)* has a Maclaurin series representation, then it is given by :

$$f(x) = f(0) + f'(0).x + \frac{f''(0)}{2!}.x^2 + ... + \frac{f^{(n)}(0)}{n!}.x^n$$

$$= 0 + 0x + \frac{0}{2!}.x^2 + ... + \frac{0}{n!}.x^n$$

which implies that f(x) = 0 throughout an interval containing 0. However this contradicts the definition of *f*. Consequently, *f(x)* does not have a Maclaurin series representation.

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Example **: 48**

Use the power series representation for e^x to find approximate value for $\int_0^{0.5} e^{-x^3}dx$

Solution

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$. Replace *x* by x^3 to obtain,

$e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + ...$. Then

$$\int_0^{0.5} e^{-x^3}dx \simeq \int_0^{0.5}\left[1 - x^3 + \frac{x^6}{2!}\right]dx \simeq \left[x - \frac{x^4}{4!} + \frac{x^7}{14}\right]_0^{0.5}$$

$$\simeq 0.5 - \frac{(0.5)^4}{4} + \frac{(0.5)^7}{14} \simeq 0.484933035$$

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Example **: 49**

Use the power series representation for e^x to find approximate value for $\int_0^{0.1} e^{-x^2}dx$.

Solution

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...$$

$$\int_0^{0.1} e^{-x^2} = \int_0^{0.1}\left(1 - x^2 + \frac{x^4}{2!} - ...\right)dx = x - \frac{x^3}{3} + \frac{x^5}{10}\Bigg|_0^{0.1} = 0.099667666$$

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***Example* : 50**

Find the Maclaurin series of the function $f(x) = \sin x$, then evaluate, $\int_0^{0.1} \sin x^2 \, dx$

Solution

$f(x) = \sin x, \quad f(0) = 0, \quad a_0 = 0$

$f'(x) = \cos x, \quad f'(0) = 1, \quad a_1 = 1/1!$

$f''(x) = -\sin x, \quad f''(0) = 0, \quad a_2 = 0$

$f'''(x) = -\cos x, \quad f'''(0) = -1, \quad a_3 = -1/3!$

$f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0, \quad a_4 = 0$

$f^{(5)}(x) = \cos x, \quad f^{(5)}(0) = 1, \quad a_5 = 1/5!$

Then $\quad \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots .$

$$\int_0^{0.1} \sin x^2 \, dx = \int_0^{0.1} \left(\frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots \right) dx = \left. \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} \right|_0^{0.1} = 0.00033333$$

***Example* : 51**

Find Maclaurin's series for $\cos x$, then approximate $\int_0^{0.5} \cos x^2 dx$

Solution

$f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x$

$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1$

$a_0 = 1 \quad a_1 = 0, \quad a_2 = \frac{-1}{2!}, \quad a_3 = 0, \quad a_4 = \frac{1}{4!}$

Maclaurin series for cos *x*: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots .$

Replace x by x^2 to obtain Maclaurin series for cos x^2 **as**

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^6}{4!} + \ldots .$$

Then, $\int_0^{0.5} \cos x^2 dx \simeq \int_0^{0.5} \left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} \right] dx \quad \simeq \left[x - \frac{x^5}{5 \times 2!} + \frac{x^9}{(9) \times 4!} \right]_0^{0.5}$

$$\simeq \left[0.5 - \frac{(0.5)^5}{10} + \frac{(0.5)^9}{9 \times (24)} \right] \simeq 0.49688 \ .$$

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Frequently-Used Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < 1.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$\cos x = x - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n - \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 \prec x \prec 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \dots = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}, \quad |x| \le 1$$

Binomial Series

$$(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k, \quad |x| \prec 1$$

where $\binom{n}{0} = 1$ and $\binom{n}{k} = \dfrac{n(n-1)\dots(n-k+1)}{k!}$ for $k \ge 1$.

Exercise (4– 6)

(I) Find the Maclaurin series of the following functions

(1) $f(x) = e^{x}$, (2) $f(x) = e^{-2x}$, (3) $f(x) = e^{3x}$

(4) $f(x) = e^{x^3}$, (5) $f(x) = \sin x$, (6) $f(x) = \cos x$.

(7) $f(x) = \sin 2x$, (8) $f(x) = \cos 3x$, (9) $f(x) = x \sin 2x$

(10) $f(x) = \cos x^2$, (11) $f(x) = \sin^2 x$, (12) $f(x) = \sin^{-1} x$

(II) Find the Maclaurin series of the following functions.

(1) $f(x) = e^{-x}$, (2) $f(x) = \cos x$, (3) $f(x) = \tan^{-1} x$, (4) $f(x) = \cos^{-1} x$.

Then show that the Maclaurin series represents these functions for all real number *x*.

(III) Find the Taylor series for the following functions at the indicated points

(1) $f(x) = \sin x$; $c = \pi/4$, and at $c = \pi/6$

(2) $f(x) = \sin 2x$; $c = \pi/3$, and at $c = \pi/6$

(3) $f(x) = 1/x$; $c = 3$. (4) $f(x) = 1/x^2$; $c = 1$.

(5) $f(x) = \cos x$; $c = \pi/3$. (6) $f(x) = \cos 3x$; $c = \pi/6$.

(7) $f(x) = e^{-x}$; $c = -2$ (8) $f(x) = e^{-2x}$; $c = -1$

(9) $f(x) = x\, e^{x}$; $c = 1$ (10) $f(x) = \csc x$; $c = 2\pi/3$

(11) $f(x) = \tan x$; $c = \pi/4$ (12) $f(x) = \sin^{-1} x$; $c = \pi/3$

(IV) (a) Find the power series representation for $f(x) = \dfrac{1}{1+x}$ if: $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$

(b) Use the series in part (a) to approximate $\ln 1.2$ ln 1.2 and $\ln 0.9$ to three decimal places and compare the approximation with that obtained using a calculator.

(V) Use the first three non-zero terms of Maclaurin series for $\tan^{-1} x$ to approximate the following,

(1) $\tan^{-1}(0.1)$, (2) $\tan^{-1}(0.5)$, (3) $\int_0^{0.1} \tan^{-1} x^2\, dx$, (4) $\int_0^{0.3} \tan^{-1} x^2\, dx$

(VI) Use the first three non-zero terms of Maclaurin series to approximate the following,

(1) $\int_0^{0.5} \cos x^2 \mathrm{d}x$, (2) $\int_0^{0.1} \sin x^2 \mathrm{d}x$, (3) $\int_0^{0.5} \mathrm{e}^{-x^3} \mathrm{d}x$

(4) $\int_0^{0.2} \tan x^2 \mathrm{d}x$, (5) $\int_0^{1/3} \frac{1}{1+x^6} \mathrm{d}x$, (6) $\int_0^{0.2} \frac{1}{1+x^4} dx$

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HIGHER TECHNOLOGICAL INSTITUTE
Tenth Of Ramadan City
Department Of Computer Science

Subject :(MTH002) model Exam1(midterm) ______________

Question 1:

a- Find an equation of the parabola and the focus that satisfied the given condition V(4,- 2) , directrix d: y = 5

b- Show that the following function $f(x,y) = e^{-2y}\cos 2x$ **satisfy the *two-dimensional Laplace* equation:**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Question 2:

a- For the curve

1- Find an equation of the tangent to the curve, when t = 0.

2- Find ——

b-Consider the following parametric curve:

Compute the arc length of this curve

Question 3:

a) Compute the area bounded by the curve –

b)Show that $\lim_{(x,\ y)\to(0,\ 0)} \frac{x}{x^2 - y}$ **does not exist.**

c)Describe and sketch the graph of the polar equations, r = $\frac{6}{4-4\cos\theta}$.

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Subject :(MTH002)   model Exam2(midterm)   ________________

Question 1

a)Describe and sketch the conic sections:

b)Show that $\lim_{(x,\ y)\to(0,\ 0)} \dfrac{x\ \ y^2}{x^2\ +\ y^4}$ **does not exist**

Question 2

a)Find the area of the surface generated by revolving of the curve C about x-axis:

b)Describe and sketch the graph of the polar equations, r = $\dfrac{6}{4-\cos\theta}$**.**

Question3:

a) **Compute** and **sketch** the area bounded by the curve

b) Find $\dfrac{\partial z}{\partial u}$ **and** $\dfrac{\partial z}{\partial v}$ **for the following function,**

$z\ =\ e^x \ln y\ \ ;\qquad x\ =\ u^2 - 5v\ ,\qquad y\ =\ v^2 - 2u$

c)For the cardioid $r\ =\ 3-3\cos\theta$ **with** $0 \le \theta \le 2\pi$**, find,**

1- The slope of the tangent line at $\theta = \pi/6$**.**

2-The points at which the tangent is horizontal or vertical.

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Subject :(MTH002)   Final Exam

Answer of the following questions:

[Q1] **[10 marks]**

a) Discuss and **sketch** the graph of the equation $4x^2+9y^2+64x-18y-71=0$. [4 marks]

b) Determine whether the following series is converges or diverges [6 marks]

1) $\sum_{n=1}^{\infty}\frac{\sqrt[3]{n^2+4}}{2n^2-n-1}$ **2)** $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^{2n}}$ **3)** $\sum_{n=1}^{\infty}\frac{4^n n!}{(2n)!}$

[Q2] **[10 marks]**

a) Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{10^n x^n}{n!}$ [6 marks]

b) Find the tangent plane and normal line to the surface $x^2+2xy-y^2+z^2=0$ at point $P(1,1,\sqrt{2})$. [4 marks]

[Q3] **[10 marks]**

a) Find the area of the region that is inside the cardioids $r=2+2\sin\theta$ and outside the circle $r=3$. [4 marks]

b) Let C be the curve with parameterization: $x=4t^2,\ y=t^3-12t$

(i) Find the equations of the tangent and normal lines to C at $t=1$.

(ii) For what values of t is the tangent line horizontal or vertical? [6 marks]

[Q4] **[10 marks]**

a) Find the three non-zero terms of Maclaurin series of $f(x)=\sin^{-1}x$ to **approximate** $\int_0^1 \sin^{-1}x\ dx$. [5 marks]

b) Find the Taylor series of $f(x)=\frac{1}{x}$ at $c=3$. [5 marks]

[Q5] **[10 marks]**

a)Find $\frac{dy}{dx}$ if $y=f(x)$ is determined implicitly by $\ln y^3+e^{xy}=\sinh(x+y)$. [4 marks]

b)Find all local maxima, local minima and the saddle points of the equation $f(x,y)=x^2-3xy-y^2+2y-6$ [6 marks]