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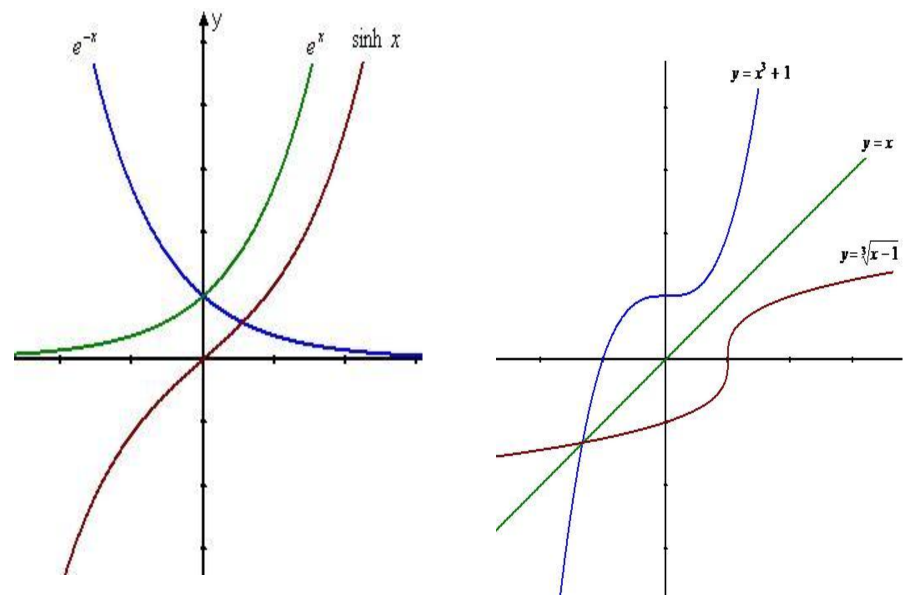

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Chapter (1)

System of linear equations

1.1. Matrices

A matrix over a field F (Field of scalars) is a rectangular array of scalars a_{ij} of the form,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (1.1)$$

Matrices will usually be denoted by a capital letters and the elements by a small letters as,

$$A = [a_{ij}], \qquad i = 1, 2, 3, \ldots, m, \qquad j = 1, 2, 3, \ldots, n. \qquad (1.2)$$

The m-horizontal n-tuples $(a_{11}, a_{12}, \ldots, a_{1n})$, $(a_{21}, a_{22}, \ldots, a_{2n})$, . . . , $(a_{m1}, a_{m2}, \ldots, a_{mn})$, are the rows of the matrix and the n-vertical m-tuples,

$$\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ a_{m1} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ a_{m2} \end{bmatrix}, \; . \; . \; . \; . \; , \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ a_{mn} \end{bmatrix},$$

are its columns. The element a_{ij} is called the ij-component, appears in the *ith* rows and *jth* column.

The matrix A_{mxn} is a matrix with m rows and *n* columns and is called *m* by *n* matrix. The pair of numbers *(m, n)* is called its size or shape.

1.1.1 Basic Definitions

(1) <u>Equality of matrices</u>: Two matrices A and B are said to be equal if A and B have the same size and the corresponding elements are equal, i.e. if $A = [a_{ij}]$ and $B = [b_{ij}]$ with $a_{ij} = b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$.

(2) <u>Addition of matrices</u>: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be of the same size. The sum of A and B , written A + B , is the matrix obtained by adding corresponding elements of A and B, that is,

$$A + B = [a_{ij} + b_{ij}] = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ . & . & . & . \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}. \qquad (1.3)$$

(3) <u>Subtraction of matrices</u>: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be of the same size. The subtraction of the matrices , written A - B , is the matrix obtained by subtracting the corresponding elements of A and B, that is,

$$A - B = [a_{ij} - b_{ij}] = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \dots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \dots & a_{2n}-b_{2n} \\ . & . & . & . \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \dots & a_{mn}-b_{mn} \end{bmatrix}. \qquad (1.4)$$

(4) <u>Scalar multiple of a matrix</u>: Let $A = [a_{ij}]$ and a scalar $k \in F$, then the product kA is the matrix obtained by multiplying each element of A by k:

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ . & . & . & . \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}. \qquad (1.5)$$

(5) <u>Additive inverse of a matrix</u>: Let $A = [a_{ij}]$, then -A is the matrix obtained by replacing the elements of A by their additive inverses, that is,

$$-A = -[a_{ij}] = [-a_{ij}] = \begin{bmatrix} -a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & -a_{22} & \dots & -a_{2n} \\ . & . & . & . \\ -a_{m1} & -a_{m2} & \dots & -a_{mn} \end{bmatrix}. \qquad (1.6)$$

(6) <u>The zero matrix</u>: It is the matrix whose elements are zero, $O = [a_{ij} = 0]$

$$O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Theorem (1). Let V be the set of all $m \times n$ matrices over a field F. Then for any matrices $A, B, C \in V$ and any scalars $k_1, k_2 \in F$,

(i) $(A+B)+C = A+(B+C)$. (ii) $A + O = A$.

(iii) $A + (-A) = O$. (iv) $A + B = B + A$.

(v) $k_1(A + B) = k_1 A + k_1 B$. (vi) $(k_1+k_2)A = k_1 A + k_2 A$.

(vii) $(k_1 k_2) A = k_1 (k_2 A)$. $(viii)$ $1.A = A$, $0.A = O$.

Example (1). Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix}$. Then,

$$A + B = \begin{bmatrix} 1+3 & -2+0 & 3+2 \\ 4-7 & 5+1 & -6+8 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 5 \\ -3 & 6 & 2 \end{bmatrix}.$$

$$A - B = \begin{bmatrix} 1-3 & -2-0 & 3-2 \\ 4+7 & 5-1 & -6-8 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 1 \\ 11 & 4 & -14 \end{bmatrix}.$$

$$3A = \begin{bmatrix} 3(1) & 3(-2) & 3(3) \\ 3(4) & 3(5) & 3(-6) \end{bmatrix} = \begin{bmatrix} 3 & -6 & 9 \\ 12 & 15 & -18 \end{bmatrix}.$$

$$2A - 3B = \begin{bmatrix} 2 & -4 & 6 \\ 8 & 10 & -12 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 6 \\ -21 & 3 & 24 \end{bmatrix} = \begin{bmatrix} -7 & -4 & 0 \\ 29 & 7 & -36 \end{bmatrix}.$$

1.1.2 Matrix Multiplication

Let $A = [a_{ij}]$ be a matrix of size $m \times n$ and $B = [b_{jk}]$ be a matrix of size $n \times p$ (that is the number of columns of A equals the number of rows of B). Then the product AB is the $m \times p$ matrix $C = [c_{ik}]$ where,

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{in} . \qquad (1.7)$$

i.e. to find the element c_{ij} (the element in row i and column j of AB) single out row i from the matrix A and column j from the matrix B, and multiply the corresponding elements from the row and column together and then add up the resulting products.

Example (2).

$$(i)\ \begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}5&6&7\\8&9&10\end{bmatrix}=\begin{bmatrix}1*5+2*8&1*6+2*9&1*7+2*10\\3*5+4*8&3*6+4*9&3*7+4*10\end{bmatrix}$$

$$=\begin{bmatrix}21&24&27\\47&54&61\end{bmatrix}.$$

$$(ii)\ \begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&1\\0&2\end{bmatrix}=\begin{bmatrix}1*1+2*0&1*1+2*2\\3*1+4*0&3*1+4*2\end{bmatrix}=\begin{bmatrix}1&5\\3&11\end{bmatrix}.$$

$$(iii)\ \begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}1*1+1*3&1*2+1*4\\0*1+2*3&0*2+2*4\end{bmatrix}=\begin{bmatrix}4&6\\6&8\end{bmatrix}.$$

$$(iv)\ \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3&4\end{bmatrix}=\begin{bmatrix}1*3&1*4\\2*3&2*4\end{bmatrix}=\begin{bmatrix}3&4\\6&8\end{bmatrix}.$$

$$(v)\ \begin{bmatrix}3&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3*1+4*2\end{bmatrix}=\begin{bmatrix}11\end{bmatrix}.$$

$$(vi)\ \begin{bmatrix}1&-1\\1&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

From (ii) and (iii) from the above example, we note that,

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&1\\0&2\end{bmatrix}\neq\begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3&4\end{bmatrix}\neq\begin{bmatrix}3&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$$

that is the matrix multiplication in general is not commutative, i.e. $AB \neq BA$.

Note that : The definition of matrix multiplication requires that the number of columns of the first matrix A be the same as the number of rows of the second matrix B in order to form the product AB. If this condition is not satisfied, the product is undefined.

Theorem (2).

$(i)\ (A\ B)C = A\ (B\ C)$ **(**Associative law)

$(ii)\ A\ (B+C) = AB + AC$ (Left distribution law)

$(iii)\ (A+B)\ C = AC + BC$ (Right distribution law)

$(iv)\ k\ (A\ B) = (k\ A)\ B = A\ (k\ B)$ where k is a scalar .

We assume that the sums and products in this theorem are defined.

Transpose of a Matrix.

The transpose of a matrix *A*, denoted by A^T, is the matrix obtained by writing the rows of *A*, in order, as columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

In other words, if $A = [a_{ij}]$ is a matrix of size mxn, then $A^T = [a^T_{ij}]$ is the *nxm* matrix, where $a^T_{ij} = a_{ji}$ for all i and j .

Note that the transpose of a row vector is a column vector and vice versa.

The transpose operation on matrices satisfies the following properties:

Theorem (3).

$(i)\ (A + B)^T = A^T + B^T$ $\qquad (ii)\ (A^T)^T = A$

$(iii)\ (AB)^T = B^T A^T$ $\qquad (iv)\ (kA)^T = k A^T \ (k \ a \ scalar)$

Note that in (iii) the transpose of a product is the product of transpose, but in the reverse order.

Example (3)

Find $A B$ using the given partitioning for A and B,

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -1 & 0 & 2 \\ 0 & 3 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Solution.

$$A\ B = \begin{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \\ \begin{bmatrix} 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -4 & -2 \end{bmatrix} \\ \begin{bmatrix} 3 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 4 & -5 \\ -3 & -5 \end{bmatrix} \\ \begin{bmatrix} 6 & 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -3 & -5 \\ 6 & 7 \end{bmatrix}.$$

1.1.3 Square Matrices

A **square matrix** is a matrix with the same number of rows as columns. An *nxn* square matrix is said to be of **order n** and is called an **n-square matrix.**

The square matrix obey all the above properties of matrices (addition, multiplication, transpose, ., ., .).

Commuting matrices. Two matrices *A* and *B* are said to commute if: *AB* = *BA*.

In example (2,ii, iii) above, the two matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ are not commute, since}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

The two matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 5 & 4 \\ 6 & 11 \end{bmatrix} \text{ are commute, since}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 6 & 11 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 11 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 26 \\ 39 & 56 \end{bmatrix}.$$

Example (4). Find all matrices M_{2x2} that commute with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution. Let $M = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, then

$$AM = \begin{bmatrix} x+z & y+t \\ z & t \end{bmatrix} \quad \text{and} \quad MA = \begin{bmatrix} x & x+y \\ z & z+t \end{bmatrix}.$$

Set $AM = MA$, to obtain the four equations

$$x + z = x, \qquad y + t = x + y, \qquad z = z, \qquad t = z + t.$$

From the first or last equation, $z = 0$; from the second equation, $x = t$. Thus M is any matrix of the form,

$$\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}.$$

Diagonal, Trace and Identity matrix.

Let $A = [a_{ij}]$ be an n-square matrix. **The diagonal** (main diagonal) of A consists of the elements a_{ii} , $1 \le i \le n$, *i.e.* $a_{11}, a_{22}, \ldots, a_{nn}$.

The trace of A, written $\operatorname{tr} A$, is the sum of the diagonal elements, that is,

$$tr\,A = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}. \qquad (1.8)$$

The n-square matrix with $1's$ on the diagonal and $0's$ elsewhere, denoted by I_n or simply I, is called the **identity** (unit) matrix.

$$I = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & . & . \\ 0 & 0 & \ldots & 1 \end{bmatrix} \qquad (1.9)$$

The matrix I is similar to the scalar 1 in that for any matrix A (of the same order),

$$AI = IA = A. \qquad (1.10)$$

More generally: If B is an *(mxn)* matrix, then ,

$$BI_n = B \qquad \text{and} \qquad I_m B = B.$$

Theorem (5).

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are n-square matrices and k is a scalar. Then

(i) $tr\ (A + B) = tr\ A + tr\ B,$ (ii) $tr\ kA = k * tr\ A,$ (iii) $tr\ AB = tr\ BA.$

1.1.4 Invertible (Non-Singular) Matrices

A square matrix A is said to be invertible (or non-singular) if there exist a matrix B such that:

$$AB = BA = I ,$$

where I is the identity matrix. Such a matrix B is unique. The matrix B is called the **inverse** of A, and is denoted by A^{-1}. Observe that the above relation is symmetric, that is, if B is the inverse of A, then A is the inverse of B.

Example (5).

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I .$$

$$BA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I .$$

Thus, $A^{-1} = B$ and $B^{-1} = A$, i.e. A and B are invertible.

Note that it is enough to prove that $AB = I$ *or* $BA = I$ to show that A and B are invertible.

Example (6).

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} -11+0+12 & 2+0-2 & 2+0-2 \\ -22+4+18 & 4+0-3 & 4-1-3 \\ -44-4+48 & 8+0-8 & 8+1-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I .$$

Thus, $A^{-1} = B$ and $B^{-1} = A$, i.e. A and B are invertible.

Special Types of Square Matrices.

Diagonal Matrices.

A square matrix $D = \lfloor d_{ij} \rfloor$ is **diagonal** if its non-diagonal elements are all zero. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

The diagonal matrix may be notated as $D = diag\,(d_{11},\ d_{22}, \ldots, d_{nn})$, where some or all of d_{ii} may be zero. The above matrices may be written, respectively, as

$$diag\,(1, 3)\ , \qquad diag\,(2, 3, -1), \qquad diag\,(2, -1, 0, 3)\ .$$

Note that any two n-diagonal matrices commute.

Triangular Matrices.

A square matrix $A = [a_{ij}]$ is an **upper triangular matrix** or simply a triangular matrix if all elements below the main diagonal are equal to zero, that is $a_{ij} = 0, \quad for \quad i > j$. For example

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$

A square matrix A is an **upper triangular matrix** if all elements above the main diagonal are equal to zero, that is $a_{ij} = 0, \quad for \quad i < j$. For example

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Example (7). Find a lower triangular matrix A such that $A^2 = \begin{bmatrix} 9 & 0 \\ -15 & 4 \end{bmatrix}$.

Solution. Set $A = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}$. Then $A^2 = \begin{bmatrix} x^2 & 0 \\ (x+z)y & z^2 \end{bmatrix}$.

Since $\begin{bmatrix} x^2 & 0 \\ (x+z)y & z^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ -15 & 4 \end{bmatrix}$, we obtain,

$x^2 = 9$, $z^2 = 4$ and $(x+z)y = -15$. Thus $x = \pm 3$, $z = \pm 2$.

From the third equation,

if $x = 3$, $z = 2 \Rightarrow y = -3$, if $x = -3$, $z = -2 \Rightarrow y = 3$,

if $x = -3$, $z = 2 \Rightarrow y = 15$, if $x = 3$, $z = -2 \Rightarrow y = -15$.

Hence, we have four matrices,

$A = \begin{bmatrix} 3 & 0 \\ -3 & 2 \end{bmatrix}$, $A = \begin{bmatrix} -3 & 0 \\ 3 & -2 \end{bmatrix}$, $A = \begin{bmatrix} -3 & 0 \\ 15 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 0 \\ -15 & -2 \end{bmatrix}$

Symmetric Matrices.

A square matrix $A = [a_{ij}]$ is **symmetric** if $A^T = A$, that is, each $a_{ij} = a_{ji}$.

A square matrix $A = [a_{ij}]$ is **skew-symmetric** if $A^T = -A$, that is, each $a_{ij} = -a_{ji}$. In the skew-symmetric, the diagonal elements must be zero since $a_{ii} = -a_{ii}$.

Example (8).

$$A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 & 2 \\ -4 & 3 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

The matrix A is symmetric and the matrix B is skew-symmetric, while C is neither symmetric nor skew-symmetric.

Exercises 1.1

(1) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \end{bmatrix}$, compute,

(i) $A + B$ (ii) $3A$ (iii) $2A - 3B$

(2) Find $x, y, z,$ and w if : $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ x+y & 3 \end{bmatrix}$.

(3) Find AB and BA if exist, for the following matrices,

(i) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 1 & -3 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -2 & -2 \end{bmatrix}$.

(iii) $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$.

(iv) $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

(v) $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 0 & 6 \\ 1 & 3 & -5 & 1 \\ 4 & 1 & -2 & 2 \end{bmatrix}$.

(4) Find AA^T and $A^T A$, where $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$.

(5) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix}$

(a) Find A^2 and A^3 . (b) Find $f(A)$, where $f(x) = 2x^3 - 4x + 5$.

(c) Find $g(B)$, where $g(x) = x^2 - x - 8$.

(6) Let $A = \begin{bmatrix} 5 & 2 \\ 0 & k \end{bmatrix}$. Find all numbers k for which A is a root of the polynomial

(a) $f(x) = x^2 - 7x + 10$, (b) $g(x) = x^2 - 25$, (c) $h(x) = x^2 - 4$,

(7) Show that $(A - kI)$ and $(B - kI)$ commute for every scalar k if and only if A and B commute.

(8) For the following matrices, show that $AB = BA = I$,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \qquad B = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

(9) Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$

1.2 Determinants

Each n-square matrix $A = \lfloor a_{ij} \rfloor$ is assigned a special scalar called the determinant of A, denoted by det $[A]$ or $|A|$ or as nxn array of the scalars a_{ij} enclosed by straight lines,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

This form is called determinant of order n.

We begin with the special case of determinants of orders one, two, and three. Then we define a determinant of arbitrary order.

Determinants of orders one and two.

Determinants of orders one and two are defined as follows,

$$|a_{11}| = a_{11} . \qquad (1.11)$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \qquad (1.12)$$

Example (9). Find the determinants of the following matrices,

$$(i)\ [24] \qquad (ii)\ [-6] \qquad (iii) \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \qquad (iv) \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

Solution.

(i) $|24| = 24$ $\qquad$ (ii) $|-6| = 6$

$$(iii) \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2*3 - 1*4 = 6 - 4 = 2.$$

$$(iv) \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 2*6 - 4*3 = 12 - 12 = 0.$$

Determinants of order three.

Consider the arbitrary 3×3 matrix $A=\lfloor a_{ij}\rfloor$. The determinant of A is defined as follows,

$$\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\begin{aligned}&=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}\\ &\quad -a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}.\end{aligned}\qquad (1.13)$$

Observe that there are six products, each product consisting of three elements of the original matrix. Three of the products are plus-labeled (keep their sign) and three of the products are minus-labeled (change their sign).

These six products may be obtained by many techniques.

Method (1).

The determinant of the 3×3 matrix $A=\lfloor a_{ij}\rfloor$ may be written as,

$$\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$=a_{11}\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}\qquad (1.14)$$

$$=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}.$$

The determinant of the 3×3 matrix may be written as a linear combination of three determinants of order two whose coefficients (with alternating signs) form the first row of the given matrix A.

Note that each 2×2 matrix can be obtained by deleting, in the original matrix, the row and column containing its coefficient.

Method (2).

We can use the following diagrams to write $\det[A]$ as the sum of the products of the elements along the three plus-labeled arrows plus the sum of the negatives of the products of the elements along the three minus-labeled arrows. There is no such diagrammatic device to remember determinants of higher order.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\qquad\qquad \begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

Method (3).

We can also obtain $\det[A]$ as follows. Repeat the first and second columns of A as shown below. Form the sum of the products of the elements on the lines from left to right, and subtract from this number the products of the elements on the lines from right to left.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

We will use method (1) for the solution and the students can verify the result by methods (2) and (3).

Example (10). Find the determinants of the following matrices,

$$(i)\quad A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} \qquad\qquad (ii)\quad B=\begin{bmatrix} 1 & 3 & 2 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

Solution.

(i) $\det A = 1\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 1(2-3) - 2(4-9) + 3(2-3)$

$= 1(-1) - 2(-5) + 3(-1) = 6.$

(ii) $\det B = 1\begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 4 & -2 \\ 0 & 5 \end{vmatrix}$

$= 1(2-15) - 3(-4-0) + 2(20-0)$

$= 1(-13) - 3(-4) + 2(20) = 39.$

Determinants of arbitrary order.

Method (1) above may be generalized to find $\det[A]$ for any arbitrary order. This method will be discussed after the properties of the determinants.

<u>1.2.1 Properties of Determinants.</u>

We now list basic properties of the determinant.

<u>Theorem (7).</u> The determinants of a matrix A and its transpose A^T are equal, i.e.

$$|A| = |A^T|$$

Example (11). For the matrix A in the above example,

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}, \quad |A^T| = 1(2-3) - 2(4-3) + 3(6-3) = 6 = |A| .$$

Theorem (8). Let A be a square matrix.

(i) If A has a row (column) of zeros, then $|A| = 0$.

(ii) If A has two identical rows (columns), then $|A| = 0$.

(iii) If A is triangular, i.e. A has zeros above or below the diagonal, $|A| =$ product of diagonal elements. Thus $|I| = 1$.

Theorem (9). Let B is obtained from A by an elementary row (column) operation.

(i) If two rows (columns) of A were interchanged, then, $|B| = -|A|$.

(ii) If a row (column) of A was multiplied by a scalar k, then, $|B| = k\,|A|$.

(iii) If a multiple of a rows (column) with added to another row (column), then, $|B| = |A|$.

We now state the most useful properties in the following theorem.

Theorem (10).

(i) If A is invertible, i.e. A has an inverse A^{-1}, then $|A| \neq 0$.

(ii) If the product $A\,B$ exist, then $|A\,B| = |A|\,|B|$.

Example (12). Show that **:** $|A\,B| = |A|\,|B|$, where A and B in example (10).

Solution. From example (18), $|A| = 6$, $|B| = 39$. Then $|A||B| = 234$.

Now,

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 14 & 5 \\ 6 & 19 & 4 \\ 7 & 17 & 7 \end{bmatrix}$$

Then,

$$|AB| = 9(65) \; -14(14) + 5(-31) \; = \; 234 \; = \; |A||B| \, .$$

Example(13). Find the determinant of each of the following matrices,

$$(i) \;\; A = \begin{bmatrix} 5 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 8 \\ 8 & 5 & 4 & 4 \end{bmatrix}, \qquad (ii) \;\; B = \begin{bmatrix} 5 & 6 & 7 & 6 \\ 1 & 3 & 5 & 3 \\ 4 & 9 & 3 & 9 \\ 2 & 7 & 8 & 7 \end{bmatrix}, \qquad (iii) \;\; C = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -3 & 7 & -8 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Solution.

(i) Since A has a row of zeros, det $A = 0$.

(ii) Since the second and fourth columns of B are equal, det $A = 0$.

(iii) Since C is triangular matrix, det (C) is equal to the product of the diagonal elements. Hence det $(C) = -120$.

1.2.2 Gaussian Elimination Algorithm for Determinants

Here $A = \lfloor a_{ij} \rfloor$ is a nonzero n-square matrix with $n > 1$.

Step (1). Choose an element $a_{ij} = 1$ or, if lacking, $a_{ij} \neq 0$.

Step (2). Using a_{ij} as a pivot, apply elementary row (column) operations to put $0's$ in all the other positions in the column (row) containing a_{ij} .

Step (3). Expand the determinant by the column (row) containing a_{ij} .

Remark (1): This algorithm is usually used for determinants of order four or more. For determinants of order less than four use the previous used method.

Example(14). Use the above algorithm to find $|A|$ for he matrix,

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{bmatrix}.$$

Solution. Use a_{23} as a pivot to put $0's$ in the other positions (using row operations),

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} \quad \text{by using} \quad \begin{matrix} -2R_2 + R_1 \to R_1 \\ 3R_2 + R_3 \to R_3 \\ R_2 + R_4 \to R_4 \end{matrix}.$$

Now we expand the third column, we may neglect all terms contains 0. Thus

$$|A| = (-1)^{2+3} \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = -[1(1) + 2(-7) + 5(-5)] = 38.$$

Minors, Cofactors and Adjoint matrix.

Consider an *n*-square matrix $A = [a_{ij}]$. Let M_{ij} denote the *(n-1)*-square submatrix of A obtained by deleting its ith row and jth column. The determinant $|M_{ij}|$ is called **the minor** of the element a_{ij} of A, and we define the **cofactor** of a_{ij}, denoted by A_{ij} to be the "signed" minor:

$$A_{ij} = (-1)^{i+j} \, |M_{ij}| \; .$$

Now, we can write the matrix of cofactors of A as,

$$Cofactors \;\; A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ . & . & . & . \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{bmatrix}.$$

We can define the **adjoint** matrix, denoted by adj A, as the transpose of the cofactors of A:

$$adj \;\; A = \begin{bmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ . & . & . & . \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{bmatrix}.$$

Theorem (11). For any square matrix A, I is the identity matrix,

$$A\,.\,(adj\,A) = (adj\,A)\,.\,A = |A|\,I\,.$$

Theorem (11) gives us another method of obtaining the inverse of non-singular matrix.

Theorem (12). For any square matrix A, if $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|}(adj\;\;A).$$

Example(15). Find the inverse matrix for the matrix , $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Solutions. The cofactors of the four elements are,

$$A_{11} = +|3| = 3, \quad A_{12} = -|1| = -1, \quad A_{21} = -|5| = -5, \quad A_{22} = +|2| = 2$$

$$Cofactor\;\;of\;\;A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}, \qquad adj\;\;A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}, \quad |A| = 6 - 5 = 1.$$

Then, $$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Example(16). Find the inverse matrix for the matrix , $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$.

Solution: The cofactors of the nine elements are,

$$A_{11} = +\begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} = -11, \qquad A_{12} = -\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} = -4, \qquad A_{13} = +\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6,$$

$$A_{21} = -\begin{vmatrix} 0 & 2 \\ 1 & 8 \end{vmatrix} = 2, \qquad A_{22} = +\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = 0, \qquad A_{23} = -\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} = -1,$$

$$A_{31} = +\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = 2, \qquad A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1, \qquad A_{33} = +\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1,$$

$$Cofactor\ of\ A=\begin{bmatrix}-11 & -4 & 6\\ 2 & 0 & -1\\ 2 & 1 & -1\end{bmatrix}, \qquad adj\ A=\begin{bmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{bmatrix},$$

$$|A| = 1(-11)+2(6) = 1 .$$

Then, $$A^{-1} = \begin{bmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{bmatrix} .$$

Example(17). Find the inverse matrix for the matrix , $A = \begin{bmatrix}2 & 3 & -4\\ 0 & -4 & 2\\ 1 & -1 & 5\end{bmatrix}$.

Solutions. The cofactors of the nine elements are,

$$A_{11} = +\begin{vmatrix}-4 & 2\\ -1 & 5\end{vmatrix} = -18, \qquad A_{12} = -\begin{vmatrix}0 & 2\\ 1 & 5\end{vmatrix} = 2, \qquad A_{13} = +\begin{vmatrix}0 & -4\\ 1 & -1\end{vmatrix} = 4,$$

$$A_{21} = -\begin{vmatrix}3 & -4\\ -1 & 5\end{vmatrix} = -11, \qquad A_{22} = +\begin{vmatrix}2 & -4\\ 1 & 5\end{vmatrix} = 14, \qquad A_{23} = -\begin{vmatrix}2 & 3\\ 1 & -1\end{vmatrix} = 5,$$

$$A_{31} = +\begin{vmatrix}3 & -4\\ -4 & 2\end{vmatrix} = -10, \qquad A_{32} = -\begin{vmatrix}2 & -4\\ 0 & 2\end{vmatrix} = -4, \qquad A_{33} = +\begin{vmatrix}2 & 3\\ 0 & -4\end{vmatrix} = -8,$$

$$Cofactor\ of\ A=\begin{bmatrix}-18 & 2 & 4\\ -11 & 14 & 5\\ -10 & -4 & -8\end{bmatrix}, \qquad adj\ A=\begin{bmatrix}-18 & -11 & -10\\ 2 & 14 & -4\\ 4 & 5 & -8\end{bmatrix},$$

$$|A| = 2(-18)+1(-10) = -46 .$$

Then, $$A^{-1} = \frac{1}{-46}\begin{bmatrix}-18 & -11 & -10\\ 2 & 14 & -4\\ 4 & 5 & -8\end{bmatrix} = \begin{bmatrix}\frac{9}{23} & \frac{11}{46} & \frac{-5}{23}\\ \frac{-1}{23} & \frac{-7}{23} & \frac{2}{23}\\ \frac{-2}{23} & \frac{-5}{46} & \frac{4}{23}\end{bmatrix} .$$

<u>Exercises 1.2</u>

(1) Find the determinants of the following matrices,

$$A = \begin{bmatrix}6 & -1\\ 3 & 3\end{bmatrix}, \qquad B = \begin{bmatrix}2 & -1\\ 1 & 5\end{bmatrix}, \qquad C = \begin{bmatrix}3 & 5\\ 6 & 10\end{bmatrix},$$

$$D = \begin{bmatrix}1 & -2 & 3\\ 2 & 4 & -1\\ 1 & 5 & -2\end{bmatrix}, \qquad F = \begin{bmatrix}2 & 3 & 4\\ 0 & 6 & 7\\ 0 & 0 & 1\end{bmatrix}, \qquad H = \begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{bmatrix}, \qquad N = \begin{bmatrix} 5 & 6 & 7 & 6 \\ 1 & -3 & 5 & -3 \\ 4 & 9 & -3 & 9 \\ 2 & 7 & 8 & 7 \end{bmatrix}.$$

(2) Find the inverse matrix for the following matrices, if it exist,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & 8 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(3) Without expanding the determinant, show that,

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = 0.$$

(4) Find the inverse of the following matrices,

$$A = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 8 & -3 \\ 1 & 7 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 2 & 1 & -2 \\ 5 & 2 & -3 \\ 0 & 2 & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & 5 \end{bmatrix}.$$

1.3 Solution of Linear Systems of Equations

We are going to look at techniques for solving linear systems of equations using matrices; it is called direct methods. A matrix derived from a linear system of equations, each in standard form, is called the augmented matrix of the system. The augmented matrix corresponding to the system of n linear algebraic equations

$$
\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_{21} + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\qquad\qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{array}\right\} \qquad (1.15)
$$

is given by the $n\times(n+1)$ augmented matrix

$$
[A\ B]=\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & b_n
\end{array}\right] \qquad (1.16)
$$

Each row of the augmented matrix represents one equation of the system. In each row, the coefficient of x_1 is in the first column, the coefficient of x_2 is in the second column, the coefficient of x_i is in the $i-$ th column, and the constant is in the last column.

Equivalence and Elementary Row Operations

A matrix A is said to be ***row equivalent*** to a matrix B , written $A \sim B,$ if B can be obtained from A by a finite sequence of the following ***elementary row operations***:

$[E_1]$ (Row-interchange) : Interchange the ith row and the jth row : $R_i \leftrightarrow R_j$.

$[E_2]$ (Row-scaling) : Multiply the ith row by a nonzero scalar k : $kR_i \rightarrow R_i$, $k \neq 0$.

$[E_3]$ (Row-addition) : Replace the ith row by k times the jth row plus the ith row:

$k\,R_j + R_i \rightarrow R_i$.

To solve a system of equations using the augmented matrix, we will use matrix row transformations to convert the augmented matrix into upper triangular form.

The following matrix is an upper-triangular form of the augment matrix (1.16):

$$[A\ B]=\begin{bmatrix} 1 & d_{12} & \cdots & d_{1n} & c_1 \\ . & 1 & \cdots & d_{2n} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_n \end{bmatrix} \qquad (1.17)$$

To obtain an upper-triangular form of an augmented matrix using matrix row transformations, we begin with the augmented matrix. We then use row transformations to obtain an equivalent matrix with ones in the diagonal and zeros below the diagonal. We will proceed column by column from left to right. In each column, we will start by obtaining a one in the diagonal position; then obtain zeros below the one.

Remark: The element we aim to be one is called pivot element (pivot) and its row is called pivoting row. To reduce the round off error in the transforming the augment matrix, we may rearrange the rows (or the equations before forming the augment matrix) to have the pivot of the first row as one, if it is possible. In the next steps, it is preferable to change the row under consideration with any row below it to make the pivot equal one. If the pivot equals zero and all the elements below it are zeros, the system either does not have unique solution nor has no solution.

1.3.1 Gauss Elimination method

This method is based on transforming the augmented matrix into a triangular one; write down the corresponding linear system of equations, which can be easily solved by back substitution. In the present approach, we use the upper triangular matrix. If we have the system of equations (1), the equivalent linear system, for which the triangular matrix is the associated augmented matrix,(1.17) is

$$\left.\begin{aligned} x_1 + d_{12}x_2 + d_{13}x_3 + \ldots + d_{1n}x_n &= c_1 \\ x_2 + d_{23}x_3 + \ldots + d_{2n}x_n &= c_2 \\ &\vdots \\ x_{n-1} + d_{n-1,n}x_n &= c_{n-1} \\ x_n &= c_n \end{aligned}\right\} \qquad (1.18)$$

which can be easily solved by back substitution in which we evaluate x_n from the last equation, and then evaluate x_{n-1} using x_n and so on, that is;

$$x_n = c_n, \qquad x_{n-1} = c_{n-1} - d_{n-1,n}x_n,$$

$$x_{n-2} = c_{n-2} - (d_{n-2,n-1}x_{n-1} + d_{n-2,n}x_n), \cdots$$

In general, we can write

$$x_i = c_i - \sum_{j=i+1}^{n} d_{i,j}x_j, \quad i = n, n-1, n-2, n-3, \cdots, 2, 1 \qquad (1.19)$$

which is useful algorithm for computer programming.

<u>**Example (18)**</u>**:** Solve the equations

$$-2x + y - z = -2, \qquad 2x - y + z = 5, \qquad -x + 2y + 2z = 1$$

<u>**Solution**</u>**:** We begin by writing the system as an augmented matrix

$$[A|B] = \tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right]$$

We already have a one in the diagonal position of the first row. Now we want zeros for the elements in the first column below that one. The first zero can be obtained by multiplying the first row by (–2) and adding the results to the second row. The second zero can be obtained by adding the first row to the third row, note that the first row is unchanged. We use elementary row operations to transform this matrix into a triangular one as follows:

$$\tilde{A}=\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right]=\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right]$$

$$\tilde{A}=\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right]=\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus we have the system of linear equations

$$z = 2, \quad y - z = -3, \quad x + y - z = -2$$

Back substitution gives the solution as:

$$z = 2, \quad y = z - 3 = -1, \quad x = -y + z - 2 = 1$$

Example (19): Solve the equations

$$3x+4y+4z=7, \quad x-y-2z=2, \quad 2x-3y+6z=5$$

Solution: It is better to rearrange the given equations to be

$$x-y-2z=2, \quad 3x+4y+4z=7, \quad 2x-3y+6z=5$$

Forming the augmented matrix

$$[AB]=\tilde{A}=\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & 4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right]=\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 7 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right]$$

$$\tilde{A}=\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 1 & 10/7 & 1/7 \\ 0 & 0 & 80/7 & 8/7 \end{array}\right]=\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & 1 & 10/7 & 1/7 \\ 0 & 0 & 1 & 1/10 \end{array}\right]$$

Thus we have the system of linear equations

$$z = 0.1, \quad y + 10z/7 = 1/7, \quad x - y - 2z = 2$$

Back substitution gives: $z = .1, \ y = 0, \ x = 2.2$

Example 20.

Solve the system of linear equations given by

$x - y - 3z = -2, \qquad 3x - y - 2z = 1, \qquad 2x + 3y - 5z = -3$

Solution:

Using the Gauss elimination method, we obtain the following sequence of equivalent augmented matrices:

$$[AB] = \tilde{A} = \left[\begin{array}{ccc|c} 1 & -1 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix is in row-reduced form. Interpreting it as a system of linear equations gives

$$y - z = -1,$$
$$x - z = 0$$

a system of two equations in the three variables x, y, and z

Let's now single out one variable—say, z—and solve for x and y in terms of it. We obtain

$$x = z,$$
$$y = z - 1$$

, if we set $z = t$, where t represents some real number (called a parameter), we obtain a solution given by (t, t - 1, t). Since the parameter t may be any real number, we see that the System **has infinitely many solutions**.

Example 21:

Solve the system of linear equations given by

$x + y + z = 1, \qquad 3x - y - z = 4, \qquad x + 5y + 5z = -1$

Solution:

Using the Gauss elimination method, we obtain the following sequence of equivalent augmented matrices

$$[AB] = \tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 53 & -1 & -1 & 4 \\ 1 & 3 & 5 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & -2 \end{array}\right]$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Observe that row 3 in the last matrix reads $0x + 0y + 0z = -1$ that is, 0=1!

We therefore conclude that the System is inconsistent and has no solution.

1.3.2 Gauss – Jordan Elimination method

Another version *of Gauss Elimination* method is *Gauss – Jordan Elimination* method. In this method, we use the matrix row transformations to convert the augmented matrix into unitary form $[I|D]$ where $D=[d_1, d_2, \cdots, d_n]^T$. Then we obtain the solution directly as $x_1 = d_1, x_2 = d_2, \cdots, x_n = d_n$.

Example (22): Solve the equations by Gauss Jordan method

$$x+y+z=0, \quad x-2y+2z=4, \quad x+2y-z=2$$

Solution: We begin by writing the system as an augmented matrix

$$[A|B] = \tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 1 & 4 \end{array}\right]$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Thus the solution is $x=4, \ y=-2, \ z=-2$

Gaussian Elimination Algorithm for Inverse. This algorithm either finds the inverse of an n-square matrix A or determines that A is not invertible.

Step 1. From the nx2n matrix $M = [A \vdots I]$; that is, A is in the left half of M and the identity matrix I is in the right half of M.

<u>Step 2</u>. Row reduce *M* to echelon form. If the process generates a zero row in the *A*-half of *M*, **STOP** (*A* is not invertible). Otherwise, the *A*-half will assume triangular form.

<u>Step 3</u>. Further row reduce *M* to the row canonical form $M = [I \vdots B]$, where *I* replaced A in the left half of the matrix.

<u>Step 4</u>. Set $A^{-1} = B$.

Example (21). Find the inverse matrix of $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

Solution. First we form the block matrix $M = [A \vdots I]$ and reduce *M* to echelon form,

$$M = \left[\begin{array}{cc:cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \; R_1 \leftrightarrow R_2; \quad M \sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right] \; -2R_1 + R_2 \to R_2;$$

$$M \sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \; \begin{array}{l} 3R_2 + R_1 \to R_1 \\ -R_2 \to R_2 \end{array}; \quad M \sim \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right] = [I \vdots A^{-1}].$$

Then,

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

Example (23). Find the inverse matrix of $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$.

Solution. First we form the block matrix $M = [A \vdots I]$ and reduce *M* to echelon form,

$$M = \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \end{array}; \quad M \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

$$R_2 + R_3 \to R_3; \quad M \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right] \quad \begin{array}{l} -R_3 + R_2 \to R_2 \\ 2R_3 + R_1 \to R_1 \end{array};$$

$$M \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right] \begin{array}{l} -R_2 \to R_2 \\ -R_3 \to R_3 \end{array};$$

$$M \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] = [I \vdots A^{-1}].$$

Then,

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}.$$

1.3.3 Solving the system of linear equations using matrix inverse

Now, for the linear system $Ax = b$, where A is a $n \times n$ square matrix(coefficient matrix), b is a $n \times 1$ vector(constants) and x is a $n \times 1$ vector(variables), suppose there exists a matrix A^{-1} such that $A^{-1}A = I_n \equiv n \times n$ identity matrix. Then,

$$Ax = b \xrightarrow{\text{multiply both side by } A^{-1}} A^{-1}Ax = A^{-1}b \rightarrow I_n \cdot x = A^{-1}b \rightarrow x = A^{-1}b.$$

Example(24)

$$\begin{aligned} 3x_1 + 2x_2 - 5x_3 &= 7 \\ x_1 - 8x_2 + 4x_3 &= 9 \\ 2x_1 + 6x_2 - 7x_3 &= -2 \end{aligned} \Rightarrow \begin{bmatrix} 3 & 2 & -5 \\ 1 & -8 & 4 \\ 2 & 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}.$$

$$\Rightarrow Ax = b, \text{ where } A = \begin{bmatrix} 3 & 2 & -5 \\ 1 & -8 & 4 \\ 2 & 6 & -7 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}.$$

Example(25). Solve the following system of equations by matrix inverse:

Solve for the linear system $Ax = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

Solution:

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} \Rightarrow x = A^{-1} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Example (26). Solve the following system of equations by matrix inverse:

$$\begin{cases} x+y-z=12 \\ 2x-y+2z=-3 \\ x+2y-z=6 \end{cases}$$

Solution: We have the following matrix system

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix}, \text{ check for } \det\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}=4$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ -4 & 0 & 4 \\ -5 & 1 & 3 \end{pmatrix}\begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix}=\begin{pmatrix} \frac{27}{4} \\ -6 \\ -\frac{45}{4} \end{pmatrix}$$

Exercise 1.3

1-Find the solution of the following systems of linear equations using

(i) Gauss elimination method.

(ii) Gauss-jordan elimination method.

(iii) Inverse matrix method

a) $\begin{cases} 2x-3y=5 \\ 3x-2y=10 \end{cases}$

b) $\begin{cases} x+y+z=4 \\ 3x+3y+z=7 \\ 4x+2y+z=7 \end{cases}$

2- Find the inverse of the following matrix using Gauss elimination method

$$A=\begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}, \quad C=\begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix}$$

3- Find the solution of the following systems of linear equations using Gauss -elimination method. Define the type of solution

a-

$$\begin{aligned} 2x - 3y + z &= 6 \\ x + 2y + 4z &= -4 \\ x - 5y - 3z &= 10 \end{aligned}$$

b-

$$\begin{aligned} x - 2y + 3z &= 9 \\ 2x + 3y - z &= 4 \\ x + 5y - 4z &= 2 \end{aligned}$$

Chapter (2)

Transcendental Functions

2.1 Inverse Functions

In mathematics the term **inverse** is used to describe functions that are reverses of one another in the sense that each undoes the effect of the other.

One-to-one Functions

A function is a rule that assigns a value from its range to each point in its domain. Some functions assign the same value to more than one point. Other functions never assume a given value more than once. A function that has distinct points is called one-to-one.

Definition A function $f(x)$ is **one-to-one** on a domain D if :

$$f(x_1) \neq f(x_2) \qquad whenever \qquad x_1 \neq x_2 \ .$$

Example (1) The function $f(x) = \sqrt{x}$ is one-to-one on any domain of non-negative numbers because:

$$\sqrt{x_1} = \sqrt{x_2} \quad whenever \quad x_1 \neq x_2 \, .$$

Example (2) The function $f(x) = x^3$ is one-to-one on any domain of real numbers because:

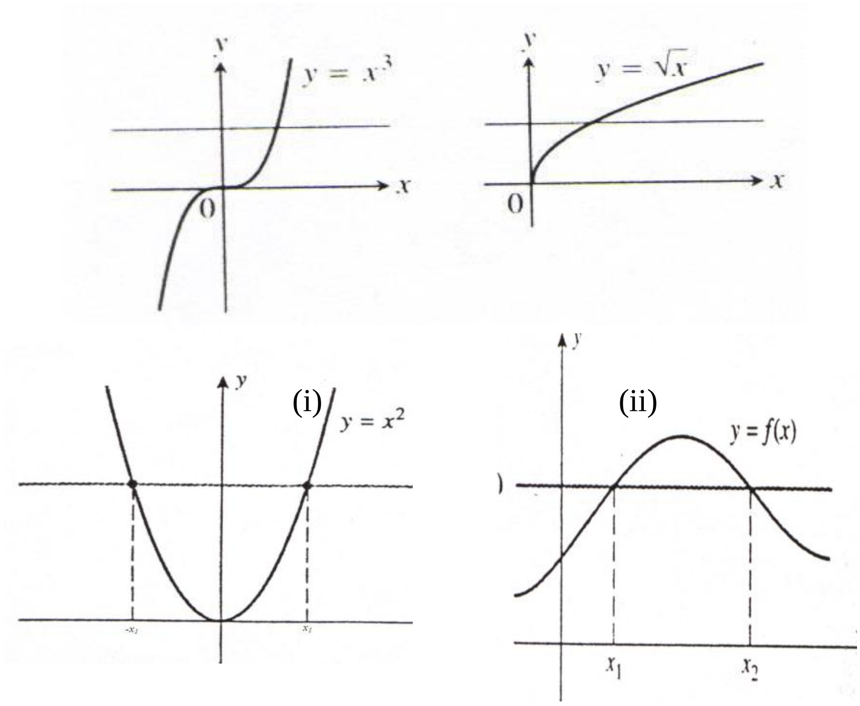
$$x_1^3 = x_2^3 \quad whenever \quad x_1 \neq x_2 \, .$$

Example (3) The functions $f(x) = x^2$ and $g(x) = \sin x$ are not one-to-one on the domain of real numbers because:

$f(-a) = f(a) = a^2$ and $g(\pi/4) = g(3\pi/4) = \ldots$

The Horizontal Line Test

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



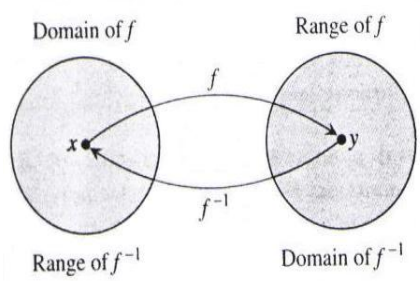
(i) (ii)

(iii) (iv)

The graphs in (i) and (ii) meets each horizontal line at most once and the graphs in (iii) and (iv) meets the horizontal line at more than one point.

> **<u>Definition</u>**
>
> If the functions *f* and *g* satisfy the two conditions
>
> $$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$
>
> $$f(g(x)) = x \text{ for every } x \text{ in the domain of } g$$
>
> then we say that *f* and *g* are inverses.

Moreover, we call *f* an inverse function of *g and g* an inverse function of *f.*
It is more convenient to write the inverse function of f as f^{-1}.



A function f has an inverse f^{-1} if it is one-to-one function.

Example (4)

Is the following functions are inverses to each other?

$$y = 2x - 2 \qquad \text{and} \qquad y = \frac{1}{2}x + 1.$$

Solution Let $f(x) = 2x - 2$ and $g(x) = \frac{1}{2}x + 1$

$$g(f(x)) = g(2x-2) = \frac{1}{2}(2x-2) + 1 = x \ .$$

$$f(g(x)) = f(\frac{1}{2}x+1) = 2(\frac{1}{2}x+1) - 2 = x \ .$$

Then the two functions $y = 2x - 2$ and $y = \frac{1}{2}x + 1$ are inverses to each other. Also, we notice that:

Domain of f^{-1} = Range of f. and

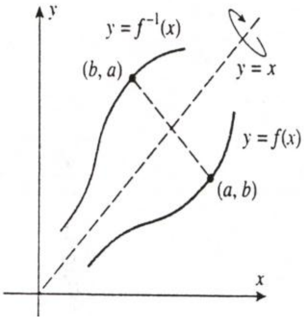
Range of f^{-1} = Domain of f.

Finding Inverses

To express the inverse function f^{-1} as a function of x, we use the following steps,

Step 1. Solve the equation $y = f(x)$ for x in terms of y.

Step 2. Interchange x and y. The resulting formula will be $y = f^{-1}$.



The graph of f^{-1} can be obtained by reflecting the graph of the function f about the line $y = x$.

Example (5)

Find the inverse of the following function

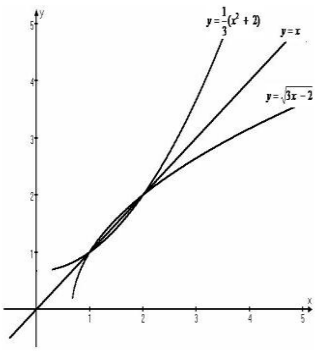
(i) $y = \frac{1}{3}(x^2 + 2)$ $\qquad (ii)$ $y = \frac{1}{2}x + 1$ $\qquad (iii)$ $y = x^3$

Solution.

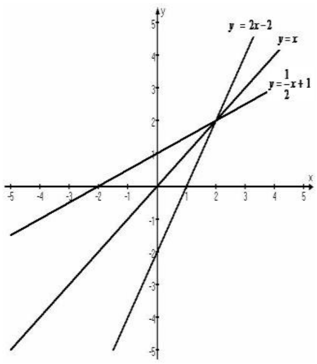
$(i)\ \ y = \dfrac{1}{3}(x^2 + 2) \Rightarrow x = \sqrt{3y - 2} \Rightarrow f^{-1}(x) = \sqrt{3x - 2}$

$(ii)\ \ y = \dfrac{1}{2}x + 1 \ \Rightarrow\ x = 2y - 2 \ \Rightarrow\ f^{-1}(x) = 2x - 2.$

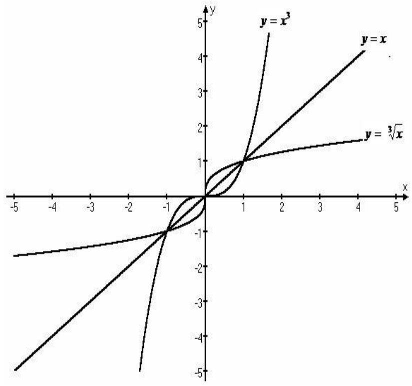
$(iii)\ \ y = x^3 \ \Rightarrow\ x = \sqrt[3]{y} \ \Rightarrow\ f^{-1}(x) = \sqrt[3]{x}\ .$



(i)



(ii)



(iii)

Exercises (2.1)

For each of the following functions, find f^{-1}, identify the domain and range of f^{-1}. Sketch the graph of f and use a reflection to sketch the graph of f^{-1}.

(1) $f(x) = x^3 + 1$

(2) $f(x) = x^5$

(3) $f(x) = x^4; \quad x \ge 0$

(4) $f(x) = 1/x^3\ ;\ x \ne 0$

(5) $f(x) = x^2 + 1\ ; \quad x \ge 0$

(6) $f(x) = x^2; \quad x \ge 0$

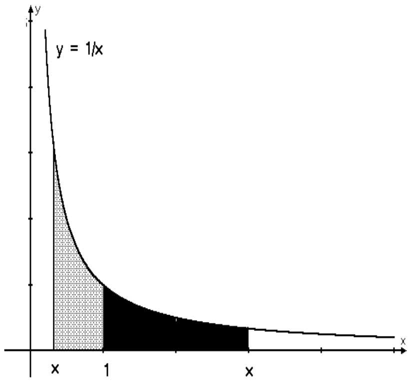
2.2 Natural Logarithmic Function

The most important function-inverse pair in mathematics and science is the pair consisting of the natural logarithmic function $\ln x$ and the exponential function e^x. The key to understanding e^x is $\ln x$, so we introduce $\ln x$ first.

Definition

The **natural logarithmic function** denoted by ln , is defined by

$$\ln\ (x) \ = \int_1^x \frac{1}{t}\ dt\ , \qquad x > 0\ .$$



If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$.

For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1. The function $\ln x$ is not defined for $x \le 0$.

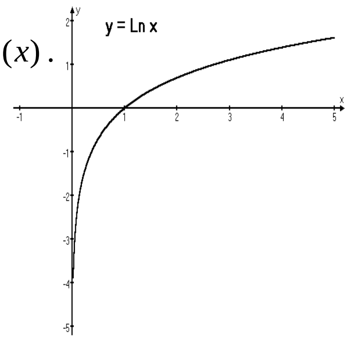
We also have,

$$\ln\ (1) \ = \int_1^1 \frac{1}{t}\ dt \ = \ 0.$$

2.2.1 Derivative of Natural Logarithmic Function

From the fundamental theorem of calculus, we know that:

$$\frac{d}{dx}\int_c^x f(t)\ dt \ = \ f(x)\ .$$



Then, $\dfrac{d}{dx}\ \ln x \ = \dfrac{d}{dx}\int_1^x \dfrac{1}{t}\ dt \ = \ \dfrac{1}{x}\ ; \qquad x \succ 0\ .$

Since the derivative of $\ln x$ is positive for every $x > 0$, then the function $y = \ln x$ is continuous and increasing throughout its domain. Moreover the second derivative is $-1/x^2$, which is negative for every $x \succ 0$. Hence the graph of $y = \ln x$ is concave-down on $(0,\ \infty)$

If $u = u(x)$, is a differentiable function of *x*, whose values are positive, so that $\ln u$ is defined. Then applying the Chain Rule,

$$\frac{d}{dx}\ln u \ = \ \frac{d}{du}\ln u .\ \frac{du}{dx} \ = \ \frac{1}{u}\ \frac{du}{dx}\ .$$

So, we obtain the derivative in the form,

$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad and \quad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

It is more convenient to write the last expression as

$$\boxed{\frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)}}$$

Example (1) Find dy/dx for the following functions,

(i) $y = \ln(x^2+1)$ $\quad (ii)$ $y = \ln(\tan x)$ $\quad (iii)$ $y = \ln(\ln(x))$

Solution

$$(i)\ \frac{dy}{dx} = \frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}.$$

$$(ii)\ \frac{dy}{dx} = \frac{d}{dx}\ln(\tan x) = \frac{1}{\tan x}\frac{d}{dx}(\tan x) = \frac{\sec^2 x}{\tan x}$$

$$(iii)\ \frac{dy}{dx} = \frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\frac{d}{dx}(\ln x) = \frac{1}{x\ln x}$$

Example (2) Find dy/dx for the following functions,

(i) $y = \ln\ln\ln(x)$ $\quad (ii)$ $y = \sin(\ln x)$

Solution

$$(i)\ \frac{dy}{dx} = \frac{d}{dx}\ln\ln\ln x = \frac{1}{\ln\ln x}\frac{d}{dx}(\ln\ln x) = \frac{1}{\ln\ln x}\frac{1}{\ln x}\frac{d}{dx}\ln x$$

$$= \frac{1}{\ln\ln x}\frac{1}{\ln x}(\frac{1}{x}) = \frac{1}{x\,(\ln x)\,(\ln(\ln x))}$$

$$(ii)\ \frac{dy}{dx} = \frac{d}{dx}\sin(\ln x) = \cos(\ln x)\frac{d}{dx}(\ln x) = \frac{\cos(\ln x)}{x}$$

<u>Properties of Logarithmic Function</u>

For any numbers $x > 0$, $y > 0$,

1. Product Rule: $\ln(xy) = \ln x + \ln y.$
2. Quotient Rule: $\ln(x/y) = \ln x - \ln y.$
3. Power Rule: $\ln x^y = y\ln x.$
4. Reciprocal Rule: $\ln(1/x) = -\ln x.$

Sometimes it is useful to use the above properties before differentiation,

Example (3) Find dy/dx for the following function,

$$y = \ln \frac{(x^2 - 1)^3 \sqrt{x+3}}{(x^3 + 1)^2}$$

Solution Use property (2),

$$y = \ln (x^2 - 1)^3 \sqrt{x+3} - \ln (x^3 + 1)^2 .$$ Use property (1),

$$y = \ln (x^2 - 1)^3 + \ln (x+3)^{1/2} - \ln (x^3 + 1)^2 .$$ Use property (3),

$$y = 3\ln (x^2 - 1) + (1/2) \ln (x+3) - 2\ln (x^3 + 1) .$$ Now differentiate

$$\frac{dy}{dx} = \frac{3(2x)}{x^2 - 1} + \frac{1}{2}\frac{1}{x+3} - \frac{2(3x^2)}{x^3+1} = \frac{6x}{x^2-1} + \frac{1}{2(x+3)} - \frac{6x^2}{x^3+1}$$

2.2.2 Logarithmic Differentiation

The derivative of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take natural logarithmic of both sides before differentiating. The following steps describe the method of solution,

Given : $y = f(x)$

Step (1) Take Logarithmic function of both sides : $\ln y = \ln f(x)$.

Step (2) Differentiate both sides w.r.t. *x*: $\frac{d}{dx} \ln y = \frac{d}{dx} (\ln f(x))$.

Step (3) Solve for dy/dx: $\frac{dy}{dx} = y \frac{d}{dx} (\ln f(x))$.

Step (4) Replace y by $f(x)$: $\frac{dy}{dx} = f(x) \frac{d}{dx} (\ln f(x))$.

We can use the result obtained in example (2) in the following example.

Example (4) Find dy/dx for the following function,

$$y = \frac{(x^2 - 1)^3 \sqrt{x+3}}{(x^3 + 1)^2}$$

Solution

Step (1) Take Logarithmic function to both sides

$$\ln\ y = \ln\ \frac{(x^2 - 1)^3\ \sqrt{x+3}}{(x^3 + 1)^2}.$$

Use the properties as we did in the above example to simplify the right term,

$$\ln\ y = 3\ln(x^2 - 1) + (1/2)\ln\ (x+3) - 2\ln\ (x^3 + 1)$$

Step (2) Differentiate both sides w.r.t. *x*:

$$\frac{d}{dx}\ln y = \frac{d}{dx}\{3\ln(x^2-1)+(1/2)\ \ln\ (x+3) - 2\ln(x^3+1)\}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{6x}{x^2-1} + \frac{1}{2(x+3)} - \frac{6x^2}{x^3+1}.$$

Step (3) Solve for dy/dx:

$$\frac{dy}{dx} = y\ \left\{\frac{6x}{x^2-1} + \frac{1}{2(x+3)} - \frac{6x^2}{x^3+1}\right\}.$$

Step (4) Replace y by $f(x)$:

$$\frac{dy}{dx} = \left(\frac{(x^2-1)^3\sqrt{x+3}}{(x^3+1)^2}\right)\left\{\frac{6x}{x^2-1} + \frac{1}{2(x+3)} - \frac{6x^2}{x^3+1}\right\}.$$

Also the logarithmic differentiation may help us to differentiate the function of the form,

$$y = [f(x)]^{g(x)},$$

Steps for differentiating the functions in the form : $y = [f(x)]^{g(x)}$

Step (1) Take Logarithm of both sides :

$$\ln y = \ln[f(x)]^{g(x)} = g(x)\ \ln[f(x)].$$

Step (2) Differentiate both sides w.r.t. *x*:

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{f'(x)}{f(x)} + g'(x)\ \ln[f(x)].$$

Step (3) Solve for dy/dx:

$$\frac{dy}{dx} = y\left\{g(x)\frac{f'(x)}{f(x)} + g'(x)\ \ln[f(x)]\right\}.$$

Step (4) Replace y by $f(x)$:

$$\frac{dy}{dx} = [f(x)]^{g(x)} \left\{ g(x)\frac{f'(x)}{f(x)} + g'(x) \ln[f(x)] \right\}.$$

Example (5) Find dy/dx for the following functions,

(i) $y = x^x$ $\qquad\qquad$ (ii) $y = (\sin x)^{\sec x}$

Solution

(i) $y = x^x$

$\ln y = \ln x^x = x \ln x$. Differentiate both sides w. r. t. x,

$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x$, then: $\frac{dy}{dx} = y\left(1 + \ln x\right) = x^x \left(1 + \ln x\right)$

(ii) $y = (\sin x)^{\sec x}$

$\ln y = \ln(\sin x)^{\sec x} = \sec x \ln \sin x$. Differentiate both sides w. r. t. x,

$\frac{1}{y}\frac{dy}{dx} = \sec x \frac{\cos x}{\sin x} + \sec x \tan x \ln(\sin x)$, then

$$\frac{dy}{dx} = y\left\{ \frac{1}{\sin x} + \sec x \tan x \ln(\sin x) \right\}$$

$$= (\sin x)^{\sec x}\left\{ \frac{1}{\sin x} + \sec x \tan x \ln(\sin x) \right\}$$

Example (6) Find dy/dx for the following functions,

(i) $y = (1+x)^{\tan x}$ $\qquad\qquad$ (ii) $y = \left(\ln x\right)^{\ln x}$

Solution

(i) $y = (1+x)^{\tan x}$

$\ln y = \ln(1+x)^{\tan x} = \tan x \ln (1+x)$. Differentiate both sides w. r. t. x,

$\frac{1}{y}\frac{dy}{dx} = \tan x \frac{1}{1+x} + \sec^2 x \ln(1+x)$, then

$$\frac{dy}{dx} = y\left\{ \frac{\tan x}{1+x} + \sec^2 x \ln(1+x) \right\}$$

$$= (1+x)^{\tan x}\left\{ \frac{\tan x}{1+x} + \sec^2 x \ln(1+x) \right\}$$

(ii) $y = \left(\ln x\right)^{\ln x}$

$\ln y = \ln \left(\ln x\right)^{\ln x} = \ln x \left(\ln \ln x\right)$. Differentiate both sides w. r. t. x,

$$\frac{1}{y}\frac{dy}{dx} = \ln x \frac{1}{\ln x}\frac{1}{x} + \frac{1}{x}\ln\ln x = \frac{1}{x}(1+\ln\ln x)$$

$$\frac{dy}{dx} = y\left\{\frac{1}{x}(1+\ln\ln x)\right\} = (1+x)^{\tan x}\left\{\frac{1+\ln\ln x}{x}\right\}$$

2.2.3 Integration and Natural Logarithmic Function

From derivatives forms we can write the following integration forms,

$$\int \frac{1}{x}\,dx = \ln|x| + c \quad \text{and} \quad \int \frac{1}{u}\,du = \ln|u| + c.$$

or in the more convenient form as,

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.}$$

Example (7) Evaluate the following integrals

$(i)\ \int \frac{x^2}{1+x^3}\,dx \qquad (ii)\ \int_0^{\pi} \frac{\sin x}{2-\cos x}\,dx \qquad (iii)\ \int \frac{1}{x\ln x}\,dx$

Solution

$$(i)\ \int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{3x^2}{1+x^3}\,dx = \frac{1}{3}\ln|1+x^3| + c.$$

$$(ii)\ \int_0^{\pi} \frac{\sin x}{2-\cos x}\,dx = \ln|2-\cos x|\Big]_0^{\pi} = \ln|3| - \ln|1| = \ln|3|\ .$$

$$(iii)\ \int \frac{1}{x\ln x}\,dx = \int \frac{1/x}{\ln x}\,dx = \ln(\ln x) + c.$$

Example (8) Evaluate the following integrals

$(i)\ \int \frac{\ln x}{x}\,dx \qquad (ii)\ \int \frac{\sin 2x}{1+\sin^2 x}\,dx \qquad (iii)\ \int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$

Solution

$$(i)\ \int \frac{\ln x}{x}\, dx = \int \frac{1}{x} \ln x \, dx = \frac{(\ln x)^2}{2} + c.$$

$$(ii)\ \int \frac{\sin 2x}{1+\sin^2 x}\, dx = \int \frac{2\sin x \cos x}{1+\sin^2 x}\, dx = \ln(1+\sin^2 x) + c$$

$$(iii) \int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\, dx = 2\int \frac{1}{(1+\sqrt{x})}\left(\frac{dx}{2\sqrt{x}}\right) = 2\ln(1+\sqrt{x}) + c$$

The natural logarithmic function may also be used to find expressions for integration of some trigonometric functions as in the following theorem.

Theorem (2.2.1)

$(i)\ \int \tan x \, dx = \ln|\sec x| + c.$

$(ii)\ \int \cot x \, dx = \ln|\sin x| + c.$

$(iii)\ \int \sec x \, dx = \ln|\sec x + \tan x| + c.$

$(iv)\ \int \csc x \, dx = \ln|\csc x - \cot x| + c.$

Proof

$$(i)\ \int \tan x \, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx = -\ln|\cos x| + c = \ln|\sec x| + c.$$

$$(ii)\ \int \cot x \, dx = \int \frac{\cos x}{\sin x}\, dx = \ln|\sin x| + c.$$

$$(iii)\ \int \sec x \, dx = \int \sec x \,\frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \ln|\sec x + \tan x| + c.$$

$$(iv)\ \int \csc x \, dx = \int \csc x \,\frac{\csc x - \cot x}{\csc x - \cot x}\, dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\, dx$$

$$= \ln|\csc x - \cot x| + c.$$

Example (9) Evaluate the following integrals,

$(i)\ \int \tan 5x \, dx$ $\qquad (ii)\ \int x \sec x^2 \, dx$

$(iii)\ \int \frac{\cot(\ln x^2)}{x}\, dx$ $\qquad (iv)\ \int x^2 \csc x^3 \, dx$

Solution

$(i)\ \int \tan 5x\ dx = \frac{1}{5}\int \tan 5x\ (5dx) = \frac{1}{5}\ \ln|\sec 5x| + c.$

$(ii)\ \int x \sec x^2\ dx = \frac{1}{2}\int \sec x^2\ (2x\,dx) = \frac{1}{2}\ln\left|\sec x^2 + \tan x^2\right| + c.$

$(iii)\ \int \frac{\cot(\ln x^2)}{x}\,dx = \frac{1}{2}\int \cot(\ln x^2)(\frac{2}{x}dx) = \frac{1}{2}\ln\left|\sin(\ln x^2)\right| + c.$

$(iv)\ \int x^2 \csc x^3\ dx = \frac{1}{3}\int \csc x^3\ (3x^2\ dx) = \frac{1}{3}\ln\left|\csc x^3 - \cot x^3\right| + c.$

Exercises (2.2)

(I) Find dy/dx for the following functions,

(1) $y = \ln(x^4 + 4)$ (2) $y = \ln(x^6 + 5)$ (3) $y = \ln|2 - 3x|$

(4) $y = \ln|x^3 - 2x|$ (5) $y = \ln|\sin x|$ (6) $y = \ln\ln\sec x$

(7) $y = \ln|\sec x + \tan x|$ (8) $y = \ln|\csc x - \cot x|$ (9) $y = [\ln(x)]^3$

(10) $y = \left[\ln|x+1|\right]^5$ (11) $y = \ln\sqrt[3]{x^2 + 1}$ (12) $y = \ln\sqrt[5]{x^4 + 3}$

(13) $y = \frac{1}{\ln x} + \ln\left(\frac{1}{x}\right)$ (14) $y = \ln x^5 + \ln\left(\frac{1}{x^5}\right)$ (15) $y = \ln\sqrt{\frac{1-x}{2-x}}$

(16) $y = \ln(1 + \ln\sqrt{x})$ (17) $y = \cos(\ln x))$ (18) $y = \tan(1 + \ln x)$

(19) $y = \ln\ \ln(\csc x^2)$. (20) $y = \frac{\ln x}{1 + \ln x}$ (21) $y = \sec(x + \ln x)$

(II) Use logarithmic differentiation to find dy/dx for the following functions,

(1) $y = (3x-2)^3(x^2 - 5)^2$ (2) $y = (2x^5 - 1)^2(x^3 + 3)^4$

(3) $y = (x^3 + 1)^4\sqrt{x^2 - 1}$ (4) $y = (x^2 - 1)^6\cot^3 x$

(5) $y = \sqrt[3]{(2x-1)\sqrt{3x^2 + 1}}$ (6) $y = \sqrt[5]{(x+3)^3(x^2 - 1)}$

(7) $y = (\sin^4 x^2)(\tan^2 x^3)$ (8) $y = (\sec^3 x^2)(\cos^2 x^3)$

(9) $y = \frac{\sqrt[4]{x^3 + 2}\ (1 + \sec x)}{(x+10)^2(x^2 + 1)^3}$ (10) $y = \sqrt[5]{(x^2 - 1)\sqrt{3x^2 - 2}}$

(11) $y = \sqrt{\dfrac{(x-3)\,(x+2)}{(x+1)(x+6)}}$

(12) $y = \dfrac{\sqrt[3]{x^2+1}\,(\sin^3 x)}{(x+1)^5\,(x^2-1)^2}$

(13) $y = (1+x)^{(1+x)}$

(14) $y = (3+\csc x)^{\cos x}$

(15) $y = (\tan x)^x$

(16) $y = (1+\ln x)^{\ln x}$

(17) $y = (\sec x)^{\sin x}$

(18) $y = (\sin + \ln x)^x$

(19) $y = (x^3 - 1)^{\csc x}$

(20) $y = (\tan x + \sec x)^{\ln x}$

(III) Evaluate the following integrals,

(1) $\int \dfrac{dx}{1+3x}$

(2) $\int \dfrac{dx}{3-x}$

(3) $\int \dfrac{x}{1-x^2}\,dx$

(4) $\int \dfrac{x+1}{x^2+2x-1}\,dx$

(5) $\int \dfrac{\ln x}{x}\,dx$

(6) $\int \dfrac{dx}{x\ln x}$

(7) $\int \dfrac{\sec^2 2x}{4-\tan 2x}\,dx$

(8) $\int \dfrac{1}{\cos^2 x(2+3\tan x)}\,dx$

(9) $\int \dfrac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$

(10) $\int \dfrac{dx}{x\sqrt{\ln x}}$

(11) $\int \dfrac{\sqrt{x}}{1+x\sqrt{x}}\,dx$

(12) $\int \dfrac{1}{x\left(1+1n\left(\sqrt{x}\right)\right)}\,dx$

(13) $\int \dfrac{\sin x - \cos x}{\sin x + \cos x}\,dx$

(13) $\int \dfrac{\tan x}{\cos x(\sec x + 5)}\,dx$

(15) $\int \dfrac{\sin x\,\cos x}{2+\cos^2 x}\,dx$

(16) $\int (\tan 3x + \sec 3x)\,dx$

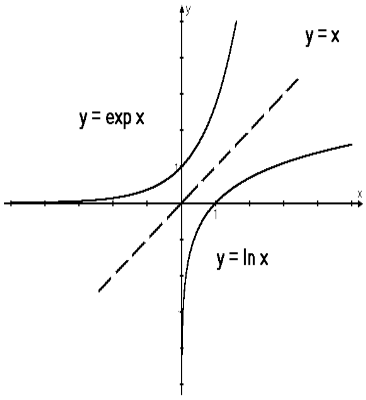
(17) $\int (\cot 5x - \csc 3x)\,dx$

(18) $\int \csc x(1-\csc x)\,dx$

(19) $\int \sec x(\sec x - 1)\,dx$

2.3. Natural Exponential Function

The Natural Logarithmic function $\ln x$ is one-to-one function with domain $(0, \infty)$ and range $(-\infty, \infty)$. Then it has an inverse function of domain $(-\infty, \infty)$ and range $(0, \infty)$. This inverse function is called the **natural exponential function** and is denoted by $\exp(x)$ or simply e^x.



Now, since $\lim_{x \to 0^+} \ln x = -\infty$

and $\lim_{x \to \infty} \ln x = \infty$, then,

$\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$,

and since $\ln(1) = 0$, then $e^0 = 1$. The function $y = e^x$ is sketched.

Since the functions $\ln x$ and e^x are inverses to each other, then

$\ln(e^x) = e^{\ln x} = x$ **and** $y = e^x$ *if and only if* $x = \ln y$.

The number e may be computed from the relation $\ln e = 1$, to obtain,

$$e = 2.7\,1828\,1828\,45\,90\,45.$$

The number e is called the base and x is called the exponent of the natural logarithmic function, *i.e.* $\ln x = \log_e x$

Properties of natural exponential function.

For any real numbers x and y,

(1) $e^x > 0$. (2) $e^{x+y} = e^x e^y$. (3) $e^{x-y} = e^x / e^y$.

(4) $e^{-x} = 1/e^x$. (5) $(e^x)^y = e^{xy} = (e^y)^x$.

2.3.1 Derivative of Natural Exponential Function

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

Consider $y = e^x$, then $\ln y = x$.

Differentiate w.r.t. *x*,

$$\frac{1}{y}\frac{dy}{dx} = 1, \quad \text{thus} \quad \frac{dy}{dx} = y = e^x$$

If u is any differentiable function of x, and by chain rule, we have,

$$\frac{d}{dx} e^x = e^x, \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

It is more convenient to write the last expression as

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$$

Example (1) Find dy/dx for the following functions,

(i) $y = x^2 e^x$ $\qquad$ (ii) $y = e^{x^2}$

(iii) $y = e^{\sqrt{x+1}}$ $\qquad$ (iv) $y = (\ln x^2)\, e^{\sin x}$

Solution

$$(i) \quad \frac{dy}{dx} = x^2 \frac{d}{dx} e^x + \frac{dx^2}{dx} e^x = x^2 e^x + 2x e^x .$$

$$(ii) \quad \frac{dy}{dx} = e^{x^2} \frac{d}{dx} x^2 = 2x\, e^{x^2} .$$

$$(iii) \quad \frac{dy}{dx} = e^{\sqrt{x+1}} \frac{d}{dx} \sqrt{x+1} = e^{\sqrt{x+1}} \frac{1}{2\sqrt{x+1}} = \frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$$

$$(iv) \quad \frac{dy}{dx} = \ln x^2 \frac{d}{dx} e^{\sin x} + e^{\sin x} \frac{d}{dx} \ln x^2$$

2.3.2 Integration of Natural Exponential Function

From the derivative formula of natural exponential function, we obtain the integral formula as,

$$\int e^x\, dx = e^x + c. \qquad \int e^u\, du = e^u + c.$$

Example (2) Evaluate the following integrals,

$(i)\ \int (e^x + 1)\, dx$ $\qquad$ $(ii)\ \int e^{2x}\, dx$

$(iii)\ \int_0^{\pi/2} e^{\sin x} \cos x\, dx.$ $\qquad$ $(iv)\ \int \frac{e^x}{e^x + 1}\, dx$

Solution

$$(i)\ \int (e^x + 1)\, dx = e^x + x + c.$$

$$(ii)\ \int e^{2x}\, dx = \frac{1}{2}\int e^{2x}\,(2\,dx) = \frac{1}{2}\, e^{2x} + c.$$

$$(iii)\ \int_0^{\pi/2} e^{\sin x}\cos x\, dx = e^{\sin x}\Big]_0^{\pi/2} = e^{\sin \pi/2} - e^{\sin 0} = e^1 - e^0 = e - 1.$$

$$(iv)\ \int \frac{e^x}{e^x + 1}\, dx = \ln(e^x + 1) + c.$$

Example (3) Evaluate the following integrals,

$(i)\ \int x\, e^{-x^2/2}\, dx$ $\qquad$ $(ii)\ \int \sec^2 2x\, e^{\tan 2x}\, dx$

$(iii)\ \int \frac{e^{2x}}{e^{2x} + 1}\, dx$ $\qquad$ $(iv)\ \int \frac{\cos\left(e^{-2x}\right)}{e^{2x}}\, dx$

Solution

$$(i)\ \int x\, e^{-x^2/2}\, dx = -\int e^{-x^2/2}\,(-x\,dx) = -e^{-x^2/2} + c.$$

Another solution

Let $u = -x^2/2, \qquad du = -x\,dx$

$$\int x\, e^{-x^2/2}\, dx = \int e^u\,(-du) = -\int e^u\, du$$

$$= -e^u + c = -e^{-x^2/2} + c.$$

$$(ii)\ \int \sec^2 2x\, e^{\tan 2x} dx = \frac{1}{2}\int e^{\tan 2x}(2\sec^2 2x\, dx) = \frac{1}{2} e^{\tan 2x} + c.$$

Another solution

Let $u = \tan 2x, \qquad du = 2\sec^2 2x\, dx$

$$\int \sec^2 2x\, e^{\tan 2x} dx = \int e^u\,\left(\frac{1}{2} du\right) = \frac{1}{2} e^u + c = \frac{1}{2} e^{\tan 2x} + c$$

$$(iii)\ \int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\ln(e^{2x}+1)+c.$$

Another solution

Let $u = e^{2x}$, $\quad du = 2\,e^{2x}\,dx$

$$\int \frac{e^{2x}}{e^{2x}+1}dx = \frac{1}{2}\int \frac{du}{u+1} = \frac{1}{2}\ln(u+1)+c = \frac{1}{2}\ln(e^{2x}+1)+c.$$

$$(iv)\ \int \frac{\cos\left(e^{-2x}\right)}{e^{2x}}\,dx = \frac{-1}{2}\int \cos\left(e^{-2x}\right)\left(-2e^{-2x}\,dx\right) = \frac{-1}{2}\sin\left(e^{-2x}\right)+c.$$

Another solution

Let $u = e^{-2x}$, $\quad du = -2\,e^{-2x}\,dx$

$$\int \frac{\cos\left(e^{-2x}\right)}{e^{2x}}\,dx = \frac{-1}{2}\int \cos u\,du = \frac{-1}{2}\sin u + c$$

$$= \frac{-1}{2}\sin\left(e^{-2x}\right)+c.$$

Exercises (2.3)

(I) Find dy/dx for the following functions,

(1) $y = e^{x}\ln x$

(2) $y = x^{3}\,e^{\sin x}$.

(3) $y = (x^{2}+1)\,e^{3x}$

(4) $y = e^{\sqrt{\sec x}}$.

(5) $y = e^{2\sin x}$

(6) $y = (x^{2}+2)^{3}\,e^{x}$

(7) $y = \ln(\cos e^{x})$

(8) $y = \sqrt{e^{\tan x}+3}$.

(9) $y = e^{-3x}\tan x$

(10) $y = \ln(\sin e^{2x})$.

(11) $y = x^{2}\,e^{\sqrt{x}}$

(12) $y = \ln(\tan(\ln x))$.

(13) $y = \dfrac{e^{x}-e^{-x}}{2}$

(14) $y = \dfrac{e^{x}+e^{-x}}{2}$

(15) $y = \dfrac{2}{e^{x}-e^{-x}}$

(16) $y = \dfrac{2}{e^{x}+e^{-x}}$

(17) $y = \sec(e^{-x^{2}})$

(18) $y = \ln x\,(\sec e^{2x})$.

(19) $y = e^{\sqrt{x+1}}\ln\sqrt{x+1}$

(20) $y = (\ln\sqrt{x})\sqrt{e^{2x}}$.

(II) Use implicit differentiation to find $\frac{dy}{dx}$,

(1) $e^{xy} + \ln x = 10$

(2) $e^{y} + \ln xy = 10$

(3) $e^{y} \cos x^2 y = 1$

(4) $e^{xy} \sin xy^2 = 10$

(5) $e^{x} \sin y + e^{y} \sin x = y$

(6) $e^{x+y} + \sin xy = x^3$

(7) $\ln(x + y) - e^{(x+y)} = 3$

(8) $\ln(xy) + e^{(x^2+y^2)} = 1$

(9) $x e^{y} + \sin(\ln y) = x$

(10) $x^2 e^{y} + \ln(\cos y) = 1$

(11) $\ln(\csc e^{x}) + e^{\csc y} = 1$

(12) $\ln(\tan e^{x}) + e^{3y} = y$

(III) Evaluate the following integrals.

(1) $\int x\, e^{x^2}\, dx$

(2) $\int x^2\, e^{x^3}\, dx$

(3) $\int (e^{x} + e^{-x})\, dx$

(4) $\int \frac{e^{\tan x}}{\cos^2 x}\, dx$

(5) $\int (e^{x} - e^{-x})^2\, dx$

(6) $\int (e^{x} + e^{-x})^2\, dx$

(7) $\int e^{x} (1 + \tan e^{x})\, dx$

(8) $\int e^{-x} (1 + \sec e^{-x})\, dx$

(9) $\int \frac{e^{(1+\sqrt{x})}}{\sqrt{x}}\, dx$

(10) $\int \frac{e^{(1+\ln x)}}{x}\, dx$

(11) $\int \frac{e^{x} + 1}{e^{x}}\, dx$

(12) $\int e^{x} \sqrt{e^{x} + 1}\, dx$

(13) $\int \frac{e^{x}}{e^{x} + 1}\, dx$

(14) $\int \sin x\ e^{\cos x}\, dx$

(15) $\int \frac{e^{x}}{\sqrt{e^{x} + 1}}\, dx$

(16) $\int \frac{e^{(1-\sqrt{x})}}{\sqrt{x}}\, dx$.

(17) $\int \frac{x\, e^{ax^2}}{e^{ax^2} + b}\, dx$

(18) $\int \frac{x\, e^{2x^2}}{e^{2x^2} + 1}\, dx$

(19) $\int \frac{\sin(e^{-2x})}{e^{2x}}\, dx$

(20) $\int \frac{\csc^2(e^{2x})}{e^{-2x}}\, dx$

2.4 General Exponential and Logarithmic Functions

2.4.1 The General Exponential Function

The general exponential function $y = a^x$ for positive number $a \neq 1$. The natural exponential function $y = e^x$ is a special case when $a = e = 2.7\,1828\,1828\,45\,90\,45$.

The general exponential function has the same properties as the natural exponential function.

Basic Properties

1. $a^x\, a^y = a^{x+y}$	2. $a^x / a^y = a^{x-y}$
3. $(ab)^x = a^x\, b^x$	4. $\left(a^x\right)^y = a^{xy}$

Derivative of general exponential function

$y = a^x$, $\quad \ln y = x \ln a$, differentiate both sides w.r.t. x,

$\dfrac{1}{y}\dfrac{dy}{dx} = \ln a$, then $\quad \dfrac{dy}{dx} = y \ln a = a^x \ln a$

$$\frac{d}{dx} a^x = a^x \ln a \qquad \text{or} \qquad \frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

In more convenient form,

$$\frac{d}{dx} a^{f(x)} = a^{f(x)} \ln a\ f'(x)$$

Example (1) Find dy/dx for the following functions,

$(i)\ y = 3^x$ $\qquad (ii)\ y = 2^{x^3}$

$(iii)\ y = 5^{\tan x}$ $\qquad (iv)\ y = 4^{x^2}\ 3^{\sin x}$

Solution

$(i)\ \dfrac{dy}{dx} = 3^x \ln 3 \qquad (ii)\ \dfrac{dy}{dx} = 2^{x^3} \ln 2\ (3x^2) \qquad (iii)\ \dfrac{dy}{dx} = 5^{\tan x} \ln 5\ (\sec^2 x)$

$(iv)\ \dfrac{dy}{dx} = 4^{x^2} \ln 4\ (2x)\ 3^{\sin x} + 4^{x^2}\ 3^{\sin x} \ln 3 \cos x \ = 4^{x^2}\ 3^{\sin x} \left(2x \ln 4 + \cos x \ln 3\right)$

Integration of general exponential function

From the derivatives formulas, we obtain the following integrations formulas

$$\int a^x \, dx = \frac{a^x}{\ln a} + c \qquad \text{and} \qquad \int a^u \, du = \frac{a^u}{\ln a} + c$$

or

$$\int a^{f(x)} f'(x)\, dx = \frac{a^{f(x)}}{\ln a} + c$$

Example (2) Evaluate the following Integral,

$(i)\ \int 3^x \, dx$ $\qquad$ $(ii)\ \int x\ 2^{x^2} \, dx$

$(iii)\ \int \cos 3x\ 5^{\sin 3x} \, dx$ $\qquad$ $(iv)\ \int \dfrac{7^{\ln x}}{x} \, dx$

Solution

$(i)\ \int 3^x \, dx = \dfrac{3^x}{\ln 3} + c\,.$

$(ii)\ \int x\ 2^{x^2} \, dx = \dfrac{1}{2}\int 2^{x^2}\ (2x\, dx) = \dfrac{1}{2}\ \dfrac{2^{x^2}}{\ln 2} + c\,.$

Another solution

Let $u = x^2, \quad du = 2x\, dx, \quad i.e. \quad \dfrac{1}{2}\, du = x\, dx$

$$\int x\ 2^{x^2} \, dx = \frac{1}{2}\int 2^u \, du = \frac{1}{2}\ \frac{2^u}{\ln 2} + c = \frac{1}{2}\ \frac{2^{x^2}}{\ln 2} + c\,.$$

$(iii)\ \int \cos 3x\ 5^{\sin 3x} \, dx = \dfrac{1}{3}\int 5^{\sin 3x}\ (3 \cos 3x\, dx) = \dfrac{1}{3}\ \dfrac{5^{\sin 3x}}{\ln 5} + c.$

Another solution

Let $u = \sin 3x, \quad du = 3 \cos 3x\, dx, \quad i.e. \quad \dfrac{1}{3}\, du = \cos 3x\, dx$

$$\int \cos 3x \; 5^{\sin 3x} \, dx = \frac{1}{3}\int 5^u \, du = \frac{1}{3}\frac{5^u}{\ln 5} + c = \frac{1}{3}\frac{5^{\sin 3x}}{\ln 5} + c.$$

$$(iv) \int \frac{7^{\ln x}}{x} \, dx = \int 7^{\ln x}\left(\frac{1}{x}\,dx\right) = \frac{7^{\ln x}}{\ln 7} + c.$$

Another solution

Let $u = \ln x$, $\qquad du = \left(\frac{1}{x}\right)dx$

$$\int \frac{7^{\ln x}}{x} \, dx = \int 7^{unx} \, du = \frac{7^u}{\ln 7} + c = \frac{7^{\ln x}}{\ln 7} + c \; .$$

2.4.2 The General Logarithmic Function

The general exponential function $f(x) = a^x$ for positive number $a \neq 1$, is one-to-one (injective) function. Its inverse function is denoted by $\log_a x$ and is called the logarithmic function of *x* with base *a*. This means that,

$$y = \log_a x \quad \textit{if and only if} \quad x = a^y.$$

The natural logarithmic function $\ln x$ is a special case of the logarithmic function when the base is *e*, that is,

$$\ln x = \log_e x.$$

The general logarithmic function has the same properties as the natural logarithmic function.

Relation between natural and general logarithmic functions. Consider $y = \log_a x$, then $x = a^y$.

Taking the natural logarithm of both sides, we obtain

$\ln x = \ln a^y = y \ln a = \ln a \log_a x$, and hence

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Derivative of general logarithmic function

Using the above relation between the general logarithmic and the natural logarithmic functions we obtain the derivative formulas as,

$$\frac{d}{dx}\log_a x = \frac{1}{\ln a}\frac{1}{x}, \qquad \frac{d}{dx}\log_a f(x) = \frac{1}{\ln a}\frac{1}{f(x)} f'(x)$$

Example (3) Find dy/dx for the following functions,

(i) $y = \log_3 x$ $\quad$ (ii) $y = \log_2 |x^2 - 4|$

(iii) $y = \log_5 |\sec x|$ $\quad$ (iv) $y = \log_7 \sqrt[3]{(2x+5)^2}$

Solution

(i) $\frac{dy}{dx} = \frac{1}{\ln 3}\frac{d}{dx}(\ln x) = \frac{1}{\ln 3}\frac{1}{x}$

(ii) $\frac{dy}{dx} = \frac{1}{\ln 2}\frac{d}{dx}\ln|x^2 - 4| = \frac{1}{\ln 2}\frac{2x}{x^2 - 4}$

(iii) $\frac{dy}{dx} = \frac{1}{\ln 5}\frac{d}{dx}\ln|\sec x| = \frac{1}{\ln 5}\frac{\sec x \tan x}{\sec x} = \frac{\tan x}{\ln 5}$

(iv) $\frac{dy}{dx} = \frac{1}{\ln 7}\frac{d}{dx}\left(\frac{2}{3}\ln|2x + 5|\right) = \frac{1}{\ln 7}\frac{2}{3}\frac{2}{2x + 5}$

Exercises 2.4

(I) In the following problems, find $\frac{dy}{dx}$ for the given functions,

(1) $y = 3^{x^5}$ $\quad$ (2) $y = 5^{(x^7+3)}$ $\quad$ (3) $y = 7^{\sec x}$

(4) $y = 2^{(\csc x + 3x)}$ $\quad$ (5) $y = 3\sin(2^x) + \sin e^x$ $\quad$ (6) $y = 2^{\ln x} + \ln 2^x$

(7) $y = 2^{e^x} + \ln 3^x$ $\quad$ (8) $y = 7^{\sin x} + 5^{\tan x}$ $\quad$ (9) $y = \tan(3^x + e^x)$

(10) $y = \cot(2^{x^3} + e^{3x})$ $\quad$ (11) $y = 10^{\ln x}$ $\quad$ (12) $y = \sec(e^{5x}) + x$

(13) $y = \log_2 (x^4 + 4)$ (14) $y = \log_7 (x^3 + 1)$ (15) $y = \log_3 |2^x - 3x|$

(16) $y = \log_2 \left|e^{3x} - x^3\right|$ (17) $y = \log_5 |\sin x + x|$ (18) $y = (\log_2 x)^4$

(19) $y = \log_2 |\sec x + x^3|$ (20) $y = \log_3 \sqrt[7]{x^3 - 1}$ (21) $y = (\log_3 x)^3$

(22) $y = \log_5 |\cos x + x^3|$ (23) $y = \log_7 \sqrt[3]{x^2 + 1}$ (24) $y = 5^{\sin x} - 3^{\ln x}$

(25) $y = \dfrac{1}{\log_3 x} + \log_3 \dfrac{1}{x}$ (26) $y = \log_3 \sqrt{x^3 - 3}$ (27) $y = \log_5 \sqrt{\dfrac{1-x}{2-x}}$

(28) $y = \dfrac{1}{\log_2 x^3} + \log_5 \dfrac{1}{x^3}$ (29) $y = 3^{\log_2 \sin x}$ (30) $y = 2^{\log_3 \tan x}$

(II) Evaluate the following integrals.

(1) $\int x\, 3^{x^2}\, dx$ (2) $\int x^2\, 5^{x^3}\, dx$ (3) $\int \left(3^x + 2^{-x}\right) dx$

(4) $\int \dfrac{2^x}{2^x + 1} dx$ (5) $\int \left(a^x - a^{-x}\right)^2 dx$ (6) $\int \left(a^{-x} + a^x\right)^2 dx$

(7) $\int 5^x \sec 5^x\, dx$ (8) $\int 3^x \tan 3^x\, dx$ (9) $\int \dfrac{2^x}{\sqrt{2^x + 9}}\, dx$

(10) $\int \dfrac{3^x}{\sqrt{3^x + 1}}\, dx$ (11) $\int \dfrac{10^{(1+\sqrt{x})}}{\sqrt{x}}\, dx$ (12) $\int \dfrac{5^{(1+\ln x)}}{x}\, dx$

(13) $\int \dfrac{7^x + 1}{7^x}\, dx$ (14) $\int 2^x \sqrt{2^x + 1}\, dx$ (15) $\int \dfrac{3^x}{3^x + 1}\, dx$

(16) $\int \dfrac{3^x + 6^x}{3^x}\, dx$ (17) $\int \dfrac{\tan(4^{-2x})}{4^{2x}}\, dx$ (18) $\int \dfrac{\sec(2^{-3x}) + 1}{2^{3x}}\, dx$

Chapter (3)

Inverse Trigonometric, Hyperbolic and Inverse Hyperbolic Functions

3.1 Inverse Trigonometric Functions

Since the trigonometric functions are not one-to-one in their natural domains, they do not have inverse functions in general. By restricting their domains, however, we obtain one-to-one functions that have the same value as the trigonometric functions and that do have inverses over these restricted domains. For example, the function $y = \sin x$ is not one-to-one on its natural domain ***R***. However, when the domain is restricted to the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, it becomes one-to-one (see Fig (3.1)).

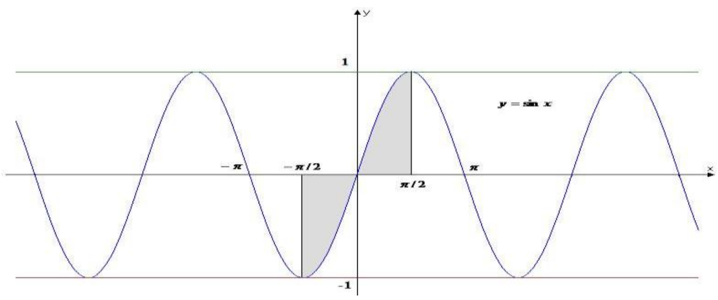


Fig. (3.1)

3.1.1 The Inverse Sine Function

Definition

The inverse **sine** function denoted by $\sin^{-1}$ x or arcsin x, is defined by ,

$$y = \sin^{-1} x \text{ or } y = \arcsin x \textit{ if and only if } x = \sin y$$

for $-1 \le x \le 1$, and $-\pi/2 \le y \le \pi/2$

We can sketch the graph of $y = \sin^{-1} x$ as in Fig (3.2) by reflecting the curve $y = \sin x$, $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ through the line $y = x$. We could also use the equation $x = \sin y$ with

$-\pi/2 \le y \le \pi/2$ to find the points on the graph $y = \sin^{-1} x$. The angles $y = \sin^{-1} x$ come from the first and fourth quadrants because $-\pi/2 \le y \le \pi/2$ [see Fig (3.3)].

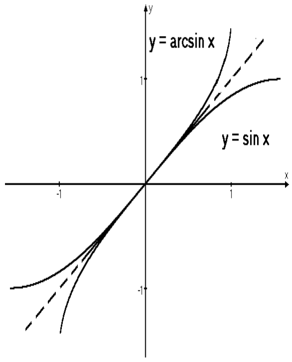


Fig. (3.2)

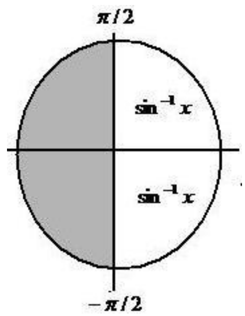


Fig. (3.3)

From the properties of inverse functions, we have :

(1) $\sin(\sin^{-1} x) = \sin(\arcsin x) = x, \quad if \ \ -1 \le x \le 1.$

(2) $\sin^{=1}(\sin x) = \arcsin(\sin x) = x, \quad if \ \ -\pi/2 \le x \le \pi/2.$

Example (1)

Evaluate the following expressions

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ (iii) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

(i) Let $y = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, then $\sin y = \frac{1}{\sqrt{2}}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Hence $y = \frac{\pi}{4}$.

(ii) Let $y = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$, then $\sin y = \frac{-1}{\sqrt{2}}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Hence $y = \frac{-\pi}{4}$.

(iii) Let $y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, then $\sin y = \frac{\sqrt{3}}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Hence $y = \frac{\pi}{3}$.

Example (2)

Find the exact value of the following expressions whenever it is defined.

(i) $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$ (ii) $\sin^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right]$ (iii) $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right]$

Solution

(i) Since $-1 \le \frac{1}{2} \le 1$, then $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{2}$.

(ii) Since $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$, then $\sin^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4}$.

(iii) Since $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\sin^{-1}(\sin\frac{2\pi}{3}) \ne \frac{2\pi}{3}$.

But, $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

3.1.2 The Inverse Cosine Function

Definition

The inverse **cosine** function denoted by $\cos^{-1} x$ or arccos x, is defined by ,

$y = \cos^{-1} x$ or $y = \arccos x$ *if and only if* $x = \cos y$

for $-1 \le x \le 1$, and $0 \le y \le \pi$.

The graph of the function $y = \cos^{-1} x$ is illustrated in Fig. (3.4). The graph may obtained in similar ways by reflecting the curve $y = \cos x$, $x \in [0, \pi]$ through the line $y = x$ or by using the equation $x = \cos y$ with $0 \le y \le \pi$. The angles $y = \cos^{-1} x$ come from the first and second quadrants because $0 \le y \le \pi$ [see Fig (3.5)].

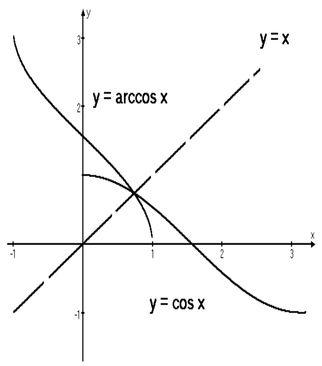


Fig. (3.4)

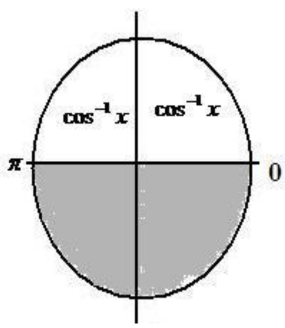


Fig. (3.5)

From the properties of inverse functions, we have the properties,

(1) $\cos(\cos^{-1} x) = \cos(\arccos x) = x, \quad if \quad -1 \le x \le 1.$

(2) $\cos^{=1}(\cos x) = \arccos(\cos x) = x, \quad if \quad 0 \le x \le \pi.$

Example (3)

Evaluate the following expressions

(i) $\cos^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(\frac{-1}{2}\right)$ (iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

(i) Let $y = \cos^{-1}\left(\frac{1}{2}\right)$, then $\cos y = \frac{1}{2}$ and $0 \le y \le \pi$.

Hence $y = \frac{\pi}{3}$.

(ii) Let $y = \cos^{-1}\left(\frac{-1}{2}\right)$, then $\cos y = \frac{-1}{2}$ and $0 \le y \le \pi$.

Hence $y = \frac{2\pi}{3}$.

(iii) Let $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$, then $\cos y = \frac{\sqrt{3}}{2}$ and $0 \le y \le \pi$. Hence $y = \frac{\pi}{6}$.

Example (4)

Find the exact value of the following expressions whenever it is defined.

(i) $\cos\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$ (ii) $\cos^{=1}\left[\cos\left(\frac{-\pi}{4}\right)\right]$

Solution

(i) Since $-1 \le \frac{-1}{2} \le 1$, then $\cos\left[\cos^{-1}\left(\frac{-1}{2}\right)\right] = \frac{-1}{2}$

(ii) Since $-\frac{\pi}{2} \le \frac{-\pi}{4} \le \frac{\pi}{2}$, then $\cos^{=1}\left[\cos\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}$

3.1.3 The Inverse Tangent Function

Definition

The inverse **tangent** function denoted by $\tan^{-1}$x or arctan x, is defined by ,

$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x \quad \textit{if and only if} \quad x = \tan y$$

for $x \in \Re$, and $-\pi/2 \le y \le \pi/2$

The graph of the function $y = \tan^{-1} x$ is illustrated in Fig. (3.6). The angles $y = \tan^{-1} x$ come from the first and fourth quadrants because $-\pi/2 \le y \le \pi/2$ [see Fig (3.7)].

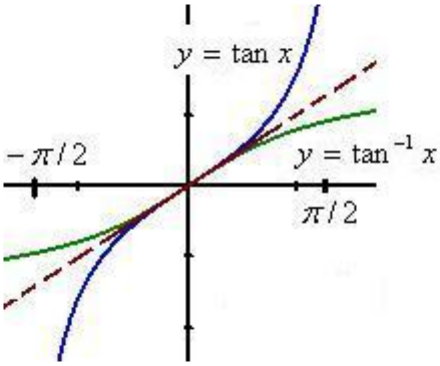


Fig. (3.6)

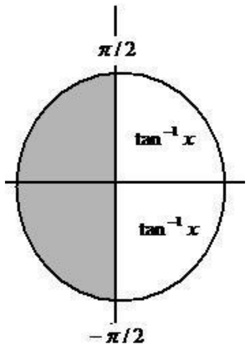


Fig. (3.7)

It has the following properties,

(1) $\tan(\tan^{-1} x) = \tan(\arctan x) = x, \quad \textit{if} \quad x \in \Re.$

(2) $\tan^{-1}(\tan x) = \arctan(\tan x) = x, \quad \textit{if} \quad -\pi/2 \le x \le \pi/2.$

Example (5)

Evaluate the following expressions

(i) $\tan^{-1}(-1)$ (ii) $\tan^{-1}(\sqrt{3})$

Solution

(i) Let $y = \tan^{-1}(-1)$, then $\tan y = -1$ and $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

Hence $y = \dfrac{-\pi}{4}$.

(ii) Let $y = \tan^{-1}(\sqrt{3})$, then $\tan y = \sqrt{3}$ and $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

Hence $y = \dfrac{\pi}{3}$

3.1.4 The Inverse Secant Function

Definition

The inverse **secant** function denoted by $\sec^{-1} x$ or arcsec x, is defined by ,

$y = \sec^{-1} x$ or $y = arc\sec x$ *if and only if* $x = \sec y$

for $|x| \geq 1$, and $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{\pi}{2}, \pi\right)$.

The graph of the function $y = \sec^{-1} x$ is illustrated in Fig. (3.8). The angles $y = \sec^{-1} x$ come from the first and third quadrants because $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{\pi}{2}, \pi\right)$ [see Fig (3.9)].

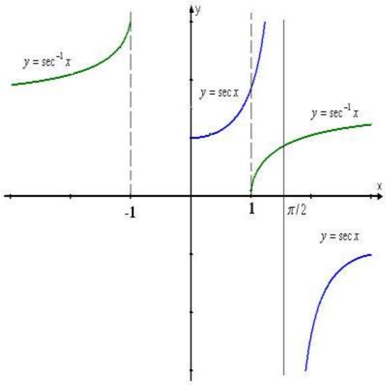


Fig. (3.8).

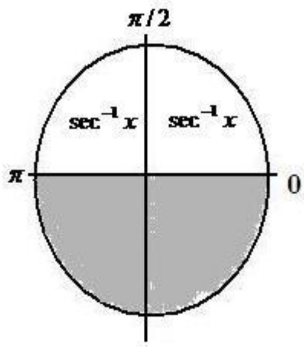


Fig. (3.9).

Remark

The domain of the inverses are chosen to satisfy the following relationships

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right), \quad \csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right), \quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right).$$

We can use these relationships to find values of $\sec^{-1} x$, $\csc^{-1} x$, $\cot^{-1} x$ on calculators that give only $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

Example (6)

Find the exact value of the following expressions whenever it is defined.

(i) $\tan\left[\tan^{-1}(100)\right]$ (ii) $\tan^{-1}\left[\tan(\pi)\right]$ (iii) $\cos^{-1}\left[\cos\left(\frac{-\pi}{4}\right)\right]$

Solution

(i) Since $100 \in \Re$, then $\tan\left[\tan^{-1}(100)\right] = 100$

(ii) Since $\pi \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then $\tan^{-1}\left[\tan(\pi)\right] = \tan^{-1}(0) = 0$

(iii) Since $\frac{-\pi}{4} \notin [0, \pi]$, then $\cos^{-1}\left[\cos\left(\frac{-\pi}{4}\right)\right] = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

Example (7)

Find the exact values of

$\sec\left[\tan^{-1}\left(\frac{2}{3}\right)\right]$, $\sin\left[\tan^{-1}\left(\frac{2}{3}\right)\right]$, $\cot\left[\tan^{-1}\left(\frac{2}{3}\right)\right]$

Solution

If we let $y = \tan^{-1}\left(\frac{2}{3}\right)$ then $\tan y = (2/3)$. We may regard y as the radian measure of an angle of a right triangle such that $\tan y = (2/3)$, as illustrated in Fig.(3.10). By the Pythagorean theorem, the hypotenuse is $\sqrt{3^2 + 2^2} = \sqrt{13}$. Referring to the triangle, we obtain,

$$\sec\left[\tan^{-1}\left(\frac{2}{3}\right)\right] = \frac{\sqrt{13}}{3}$$

$$\sin\left[\tan^{-1}\left(\frac{2}{3}\right)\right] = \frac{2}{\sqrt{13}}$$

$$\cot\left[\tan^{-1}\left(\frac{2}{3}\right)\right] = \frac{3}{2}$$

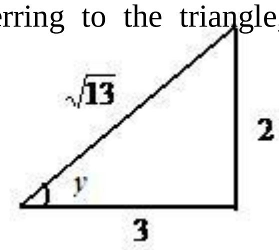


Fig.(3.10)

Example (8)

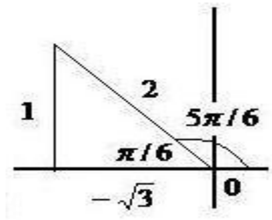
Find : $\cot\left[\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right]$

Solution

We work from inside out, using reference triangles to exhibit ratios and angles.

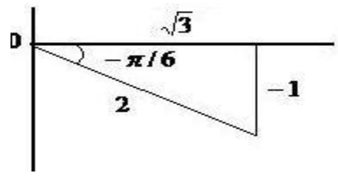
Negative values of the secant come from second-quadrant angles;

$$\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \sec^{-1}\left(\frac{2}{-\sqrt{3}}\right) = \frac{5\pi}{6}$$



Negative values of the cosecant come from fourth-quadrant angles;

$$\csc^{-1}(-2) = \csc^{-1}\left(\frac{2}{-1}\right) = \frac{-\pi}{6}$$



Then

$$\cot\left[\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right] = \cot\left[\frac{5\pi}{6} - \frac{\pi}{6}\right]$$

$$= \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}} \ .$$

Exercises (3.1)

(I) Find the exact value of the following expression whenever it is defined.

1 a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ b) $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2 a) $\sin^{-1}\left(\frac{1}{2}\right)$ b) $\cos^{-1}\left(\frac{-1}{2}\right)$ c) $\tan^{-1}(-1)$

3 a) $\sin^{-1}\left(\frac{\pi}{3}\right)$ b) $\cos^{-1}\left(\frac{\pi}{2}\right)$ c) $\sec^{-1}\left(\frac{1}{2}\right)$

4 a) $\sin^{-1}\left(\frac{2\pi}{3}\right)$ b) $\cos^{-1}\left(\frac{\pi}{3}\right)$ c) $\sec^{-1}\left(\frac{1}{3}\right)$

5 a) $\sin\left[\sin^{-1}\left(-\frac{3}{10}\right)\right]$ b) $\cos\left[\cos^{-1}\left(\frac{2}{3}\right)\right]$

c) $\tan\left[\tan^{-1}(14)\right]$ d) $\sec\left[\sec^{-1}(2)\right]$

6 a) $\sin\left[\sin^{-1}\left(\frac{2}{5}\right)\right]$ b) $\cos\left[\cos^{-1}\left(\frac{-3}{4}\right)\right]$

c) $\tan\left[\tan^{-1}(12)\right]$ d) $\sec\left[\sec^{-1}(5)\right]$

7 a) $\sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right]$ b) $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

c) $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$ d) $\sec^{-1}\left[\sec\left(\frac{\pi}{6}\right)\right]$.

8 a) $\sin^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right]$ b) $\cos^{-1}\left[\cos\left(\frac{2\pi}{3}\right)\right]$

c) $\tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right]$ d) $\sec^{-1}\left[\sec\left(\frac{\pi}{6}\right)\right]$.

9 a) $\sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ b) $\tan^{-1}\left[\sec\left(\frac{\pi}{4}\right)\right]$

c) $\tan\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$ d) $\sin\left[\tan^{-1}(-1)\right]$

(II) Rewrite as an algebraic expression in x for $x > 0$;

(1) $\sin(\tan^{-1} x)$ (2) $\tan(\sin^{-1} x)$

(3) $\tan(\cos^{-1} x)$ (4) $\cos(\tan^{-1} x)$

(5) $\sec\left(\sin^{-1}\frac{x}{3}\right)$ (6) $\sin\left(\cos^{-1}\frac{x}{2}\right)$

(7) $\cot\left(\sin^{-1}\frac{1}{x}\right)$ (8) $\tan\left(\sec^{-1}\frac{1}{x}\right)$

(III) Find the solutions of the equation that are in the given interval**.**

(1) $2\tan^2 t + 9\tan t + 3 = 0;$ $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(2) $15\cos^4 t - 14\cos^2 t + 3 = 0;$ $[0, \pi]$

3.2 Derivatives and Integrals

We consider next the derivatives and integrals of the inverse trigonometric functions and integrals that result in inverse trigonometric functions. We concentrate on the inverse sine, cosine, tangent, and secant functions.

Theorem (3.3.1)

If $u = f(x)$ is differentiable with x restricted to values for which the indicated function is defined,

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\ , \qquad \frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}\ , \qquad \frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Proof We shall consider only the special case $u = x$, since the formulas for $u = f(x)$ may then be obtained by applying the chain rule.

Let $y = \sin^{-1}x$, then $\sin y = x$. Differentiate both sides w.r.t. *x*, $\cos y\,\frac{dy}{dx} = 1$, and

hence, $\frac{d}{dx}\sin^{-1}x = \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

The formula for $\frac{d}{dx}\cos^{-1}x$ can be obtained in a similar way.

Let $y = \tan^{-1}x$, then $\tan y = x$. Differentiate both sides w.r.t. *x*, $\sec^2 y\,\frac{dy}{dx} = 1$, and

hence, $\frac{d}{dx}\tan^{-1}x = \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

Similarly we can find the derivative of $\frac{d}{dx}\sec^{-1}x$

Example (1) Find dy/dx for the following functions,

$(i)\ y = \sin^{-1}x^3 \qquad (ii)\ y = \tan^{-1}\sqrt{x+1}$

Solution

$(i)\ \frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}\frac{d}{dx}x^3 = \frac{3x^2}{\sqrt{1-x^6}}$.

$(ii)\ \dfrac{dy}{dx} = \dfrac{1}{1+(\sqrt{x+1})^2}\dfrac{d}{dx}\sqrt{x+1}$

$= \dfrac{1}{1+(x+1)}\dfrac{1}{2\sqrt{x+1}} = \dfrac{1}{2\sqrt{x+1}\,(x+2)}$

.

Example (2) Find dy/dx for the following functions,

$(i)\ y = \sec^{-1} 5x$ $\quad (ii)\ y = \sin^{-1} e^{2x}$ $\quad (iii)\ y = \tan^{-1}(\ln x)$

Solution

$(i)\ \dfrac{dy}{dx} = \dfrac{1}{|5x|\sqrt{(5x)^2-1}}\dfrac{d}{dx}(5x) = \dfrac{1}{|x|\sqrt{25x^2-1}}$.

$(ii)\ \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(e^{2x})^2}}\dfrac{d}{dx}e^{2x} = \dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$

$(iii)\ \dfrac{dy}{dx} = \dfrac{1}{1+(\ln x)^2}\dfrac{d}{dx}(\ln x) = \dfrac{1}{x[1+(\ln x)^2]}$.

We may use theorem (3.3.1) for differentiation to obtain the following integration formulas

<u>Theorem (3.3.2)</u>

(i) $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1} u + c; \qquad |u| < 1$

(ii) $\displaystyle\int \frac{1}{1+u^2}\, du = \tan^{-1} u + c; \qquad \textit{for all } u$

(iii) $\displaystyle\int \frac{1}{u\sqrt{u^2-1}}\, du = \sec^{-1} u + c; \qquad |u| > 1$

These formulas can be generalized for $a > 0$ as follows ,

$$\text{(i)} \int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + c; \qquad |u| < a.$$

$$\text{(ii)} \int \frac{1}{a^2 + u^2}\, du = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{u}{a}\right) + c; \qquad \textit{for all } u$$

$$\text{(iii)} \int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \left(\frac{1}{a}\right)\sec^{-1}\left(\frac{u}{a}\right) + c; \qquad |u| > a$$

Example (3) Evaluate the following integrals,

$(i)\ \int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}}$ $\qquad (ii)\ \int_0^1 \frac{dx}{1 + x^2}$

$(iii)\ \int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2 - 1}}$ $\qquad (iv)\ \int \frac{e^{2x}\, dx}{1 + e^{4x}}$

Solution

$$(i)\ \int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x\Big]_{1/\sqrt{2}}^{\sqrt{3}/2} = \sin^{-1}(\sqrt{3}/2) - \sin^{-1}(1/\sqrt{2}) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

$$(ii)\ \int_0^1 \frac{dx}{1 + x^2} = \tan^{-1} x\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

$$(iii)\ \int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x\Big]_{2/\sqrt{3}}^{\sqrt{2}} = \sec^{-1}(\sqrt{2}) - \sec^{-1}(2/\sqrt{3}) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

$$(iv)\ \int \frac{e^{2x}\, dx}{1 + e^{4x}} = \frac{1}{2}\int \frac{(2e^{2x}\, dx)}{1 + (e^{2x})^2} = \frac{1}{2}\tan^{-1}(e^{2x}) + c.$$

Example (4) Evaluate the following integrals,

$(i)\ \int \frac{x\, dx}{\sqrt{9 - x^4}}$ $\qquad (ii)\ \int \frac{dx}{x\sqrt{x^6 - 4}}$

$(iii)\ \int \frac{dx}{x(4 + [\ln x]^2)}$ $\qquad (iv)\ \int \frac{e^x\, dx}{e^{-2x} + e^{4x}}$

Solution

$$(i)\ \int \frac{x\, dx}{\sqrt{9 - x^4}} = \frac{1}{2}\int \frac{(2x\, dx)}{\sqrt{9 - (x^2)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right) + c.$$

$$(ii)\int \frac{dx}{x\sqrt{x^6-4}} = \frac{1}{3}\int \frac{(3x^2dx)}{x^3\sqrt{(x^3)^2-4}} = \frac{1}{6}\sec^{-1}\left(\frac{x^3}{2}\right) + c.$$

$$(iii)\int \frac{dx}{x(4+[\ln x]^2)} = \int \frac{(1/x)dx}{(4+[\ln x]^2)} = \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{2}\right) + c.$$

$$(iv)\int \frac{e^x\,dx}{e^{-2x}+e^{4x}} = \int \frac{e^{3x}\,dx}{1+e^{6x}} = \frac{1}{3}\int \frac{(3e^{3x}\,dx)}{1+(e^{3x})^2} = \frac{1}{3}\tan^{-1}(e^{3x}) + c.$$

Example (5) Evaluate the following integrals,

$(i)\ \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ $\qquad (ii)\ \int \frac{\cos(\sin^{-1}x)}{\sqrt{1-x^2}}\,dx$

$(iii)\ \int \frac{x^2}{5+x^6}\,dx$ $\qquad (iv)\ \int \frac{1}{(1+x^2)\tan^{-1}x}\,dx$

Solution

$$(i)\ \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int \sin^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right)dx = \frac{(\sin^{-1}x)^2}{2} + c.$$

$$(ii)\ \int \frac{\cos(\sin^{-1}x)}{\sqrt{1-x^2}}\,dx = \int \cos(\sin^{-1}x)\left(\frac{dx}{\sqrt{1-x^2}}\right)$$

$$= \sin(\sin^{-1}x) + c = x + c$$

$$(iii)\int \frac{x^2}{5+x^6}\,dx\ \frac{1}{3}\int \frac{3x^2}{(\sqrt{5})^2+x^6}\,dx = \frac{1}{3}\,\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x^3}{\sqrt{5}}\right) + c\ .$$

$$(iv)\int \frac{1}{(1+x^2)\tan^{-1}x}\,dx = \int \frac{d(\tan^{-1}x)}{\tan^{-1}x} = \ln\left|\tan^{-1}x\right| + c.$$

<u>Exercises (3.2)</u>

(I) Find dy/dx for the following functions,

(1) $y=\sin^{-1}\sqrt{x}$. (2) $y = x^3\sin^{-1}x$. (3) $y=\tan^{-1}(3x-5)$

(4) $y=\tan^{-1}(e^{2x})$. (5) $y=e^{-x}\sec^{-1}e^{-x}$ (6) $y=\sec^{-1}x^3$

(7) $y=x^2\tan^{-1}x^2$. (8) $y=\tan^{-1}(\ln x)$. (9) $y=\sec^{-1}\sqrt{x^2-1}$

(10) $y=\cos^{-1}e^{3x}$. (11) $y=\frac{1}{\sin^{-1}x}$ (12) $y=\sqrt{\sin^{-1}x}$

(13) $y = (1+\cos^{-1}3x)^3$ (14) $y = \sec^{-1}\sqrt{x}$ (15) $y = \ln(\tan^{-1}x^2)$

(16) $y = \tan^{-1}\sqrt[3]{x^2}$ (17) $y = 3^{\sin^{-1}x^3}$ (18) $y = \ln(\sin^{-1}e^{2x})$

(19) $y = \dfrac{\tan^{-1}x}{x^2+1}$. (20) $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

(21) $y = \cos(x^{-1}) + (\cos x)^{-1} + \cos^{-1}x$.

(II) Evaluate the following integrals,

(1) $\int \dfrac{1}{x^2+16}\,dx$ (2) $\int \dfrac{dx}{4+3x^2}$ (3) $\int_0^4 \dfrac{1}{x^2+16}\,dx$

(4) $\int_{-2}^{2} \dfrac{dx}{4+3x^2}$ (5) $\int \dfrac{x}{\sqrt{1-x^4}}\,dx$ (6) $\int \dfrac{dx}{x\sqrt{x^4-9}}$

(7) $\int_0^{\sqrt{2}/2} \dfrac{x}{\sqrt{1-x^4}}\,dx$ (8) $\int \dfrac{\sqrt{\tan^{-1}x}}{1+x^2}\,dx$ (9) $\int \dfrac{\sin x}{\cos^2 x+1}\,dx$

(10) $\int \dfrac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx$ (11) $\int \dfrac{1}{\sqrt{x}\,(1+x)}\,dx$ (12) $\int \dfrac{dx}{x\sqrt{5x^2-9}}$

(13) $\int \dfrac{dx}{\sqrt{e^{2x}-1}}$ (14) $\int \dfrac{dx}{\sqrt{1-e^{4x}}}$ (15) $\int \dfrac{\sec^2(\sec^{-1}x)}{x\sqrt{x^2-1}}\,dx$

3.3 Hyperbolic Functions

Many of the advanced applications of calculus involve the exponential expressions

$$\frac{e^x+e^{-x}}{2} \quad \text{and} \quad \frac{e^x-e^{-x}}{2}$$

Which define the hyperbolic functions. These hyperbolic functions are used to solve a variety of problems in the physical sciences and engineering.

3.3.1 Basic Definitions

There are six hyperbolic functions similar to the trigonometric functions called hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, and hyperbolic cosecant and denoted by $\sinh x$, $\cosh x$, $\tanh x$, $\coth x$, $\text{sec}h\,x$, and $\text{csc}h\,x$ respectively, [see Fig.(3.11)], and are defined as follows

Definitions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$$

$$\text{sec}h\,x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad \text{csc}h\,x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \neq 0$$

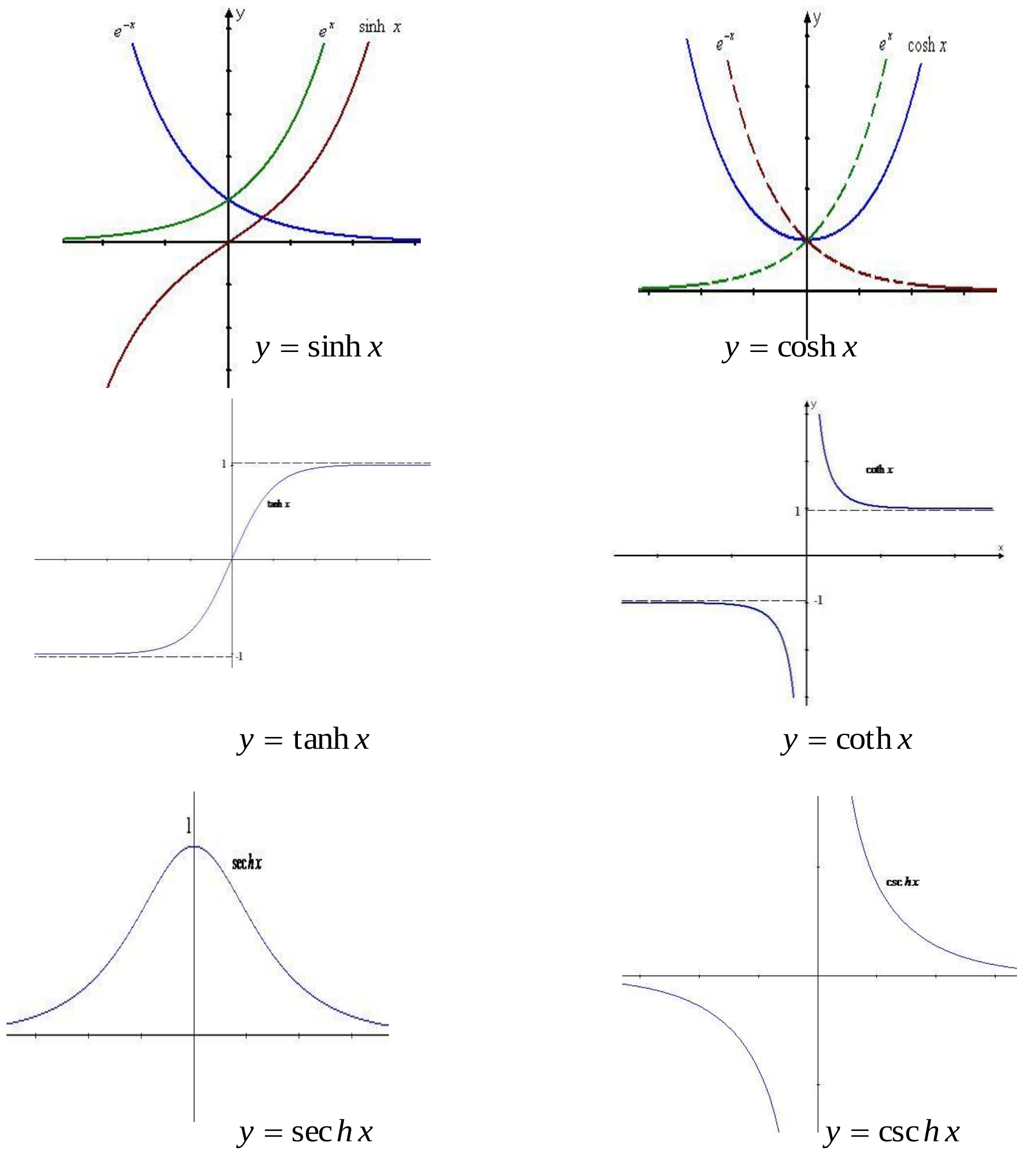


Fig.(3.11)

The hyperbolic functions has a number of identities similar to the trigonometric functions listed in the following theorem.

Theorem (3.3.1)

$(i)\ \cosh^2 x - \sinh^2 x = 1$ $\qquad$ $(ii)\ 1 - \tanh^2 x = \operatorname{sech}^2 x$

$(iii)\ \coth^2 x - 1 = \operatorname{csch}^2 x$ $\qquad$ $(iv)\ \sinh 2x = 2\sinh x \cosh x$

$(v)\ \cosh 2x = \cosh^2 x + \sinh^2 x$ $\qquad$ $(vi)\ \cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$(vii)\ \sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Example (1)

Approximate the following to four decimal places

(i) $\sinh(4)$ $\qquad$ (ii) $\operatorname{sech}(2)$

Solution

(i) $\sinh(4) = \dfrac{e^4 - e^{-4}}{2} = \dfrac{54.59815 - 0.0183156}{2} = 27.2899$

(ii) $\operatorname{sech}(2) = \dfrac{2}{e^2 + e^{-2}} = 0.2658$

3.3.2 Derivatives and Integrals

Theorem (3.3.2)

If $u = f(x)$ is differentiable , then

$(i)\ \dfrac{d}{dx}\sinh u = \cosh u \dfrac{du}{dx}$ $\qquad$ $(ii)\ \dfrac{d}{dx}\cosh u = \sinh u \dfrac{du}{dx}$

$(iii)\ \dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$ $\qquad$ $(iv)\ \dfrac{d}{dx}\coth u = -\operatorname{csch}^2 u \dfrac{du}{dx}$

$(v)\ \dfrac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$ $\qquad$ $(vi)\ \dfrac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

Proof As usual we consider only the case $u = x$.

$$(i)\ \frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$(ii)\ \frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

$(iii)\ \dfrac{d}{dx}\tanh x = \dfrac{d}{dx}\dfrac{\sinh x}{\cosh x} = \dfrac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x}$

$= \dfrac{1}{\cosh^2 x} = \operatorname{sec}h^2 x\,.$

$(iv)\ \dfrac{d}{dx}\coth x = \dfrac{d}{dx}\dfrac{\cosh x}{\sinh x} = \dfrac{\sinh x\sinh x - \cosh x\cosh x}{\sinh^2 x}$

$= \dfrac{-1}{\sinh^2 x} = -\operatorname{csc}h^2 x\,.$

$(v)\ \dfrac{d}{dx}\operatorname{sec}h\,x = \dfrac{d}{dx}\dfrac{1}{\cosh x} = \dfrac{-\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x}\dfrac{\sinh x}{\cosh x}$

$= -\operatorname{sec}h\,x\tanh x\,.$

$(vi)\ \dfrac{d}{dx}\operatorname{csc}h\,x = \dfrac{d}{dx}\dfrac{1}{\sinh x} = \dfrac{-\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x}\dfrac{\cosh x}{\sinh x}$

$= -\operatorname{csc}h\,x\coth x.$

Example (2) Find dy/dx for the following functions,

$(i)\ y = \sinh x^3$ $\qquad (ii)\ y = \operatorname{sec}h\,x^2$

$(iii)\ y = \cosh(x^2 - 1)$ $\qquad (v)\ y = \tanh(e^x)$

Solution

$(i)\ \dfrac{dy}{dx} = \cosh x^3\,\dfrac{d}{dx}x^3 = 3x^2\cosh x^3\,.$

$(ii)\ \dfrac{dy}{dx} = -\operatorname{sec}h\,x^2\tanh x^2\,\dfrac{d}{dx}x^2 = -2x\operatorname{sec}h\,x^2\tanh x^2$

$(iii)\ \dfrac{dy}{dx} = \sinh(x^2-1)\dfrac{d}{dx}(x^2-1) = 2x\sinh(x^2-1)$

$(iv)\ \dfrac{dy}{dx} = \operatorname{sec}h^2 e^x\,\dfrac{d}{dx}e^x = e^x\operatorname{sec}h^2 e^x\,.$

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Example (3) Find dy/dx for the following functions,

$(i)\ y = \tanh\sqrt{1+x^2}$ $\qquad (ii)\ y = \cosh(e^{3x})$

$(iii)\ y = \coth(\ln x)$ $\qquad (iv)\ y = \operatorname{csc}h\,(x^3)$

Solution

$(i)\ \dfrac{dy}{dx} = \operatorname{sec}h^2\sqrt{1+x^2}\,\dfrac{d}{dx}\sqrt{1+x^2} = \dfrac{x}{\sqrt{1+x^2}}\operatorname{sec}h^2\sqrt{1+x^2}\,.$

$(ii)\ \dfrac{dy}{dx} = \sinh(e^{3x})\dfrac{d}{dx}e^{3x} = 3\,e^{3x}\ \sinh(e^{3x}).$

$(iii)\ \dfrac{dy}{dx} = -\operatorname{csc}h^2(\ln x)\dfrac{d}{dx}(\ln x) = -\dfrac{1}{x}\operatorname{csc}h^2(\ln x)$

$(iv)\ \dfrac{dy}{dx} = -\operatorname{csc}h\ x^3 \coth x^3\dfrac{d}{dx}x^3 = -3x^2\operatorname{csc}h\ x^3\coth x^3$

We may use theorem (3.3.2) for differentiation to obtain the following integration formulas

Theorem (3.3.3)

$(i)\ \int \sinh u\ du = \cosh u + c.$ $\qquad (ii)\ \int \cosh u\ du = \sinh u + c.$

$(iii)\ \int \operatorname{sec}h^2 u\ du = \tanh u + c.$ $\qquad (iv)\ \int \operatorname{csc}h^2 u\ du = -\coth u + c.$

$(v)\ \int \operatorname{sec}hu\ \tanh u\ du = -\operatorname{sec}hu + c.$ $\qquad (vi)\ \int \operatorname{csc}hu\ \coth u\ du = -\coth u + c.$

Example (4). Evaluate the following integrals,

$(i)\ \int x\operatorname{sec}h^2 x^2\ dx$ $\qquad (ii)\ \int \tanh 3x\ dx$

$(iii)\ \int \sinh^2 x\ dx$ $\qquad (iv)\ \int 4e^x \sinh x\ dx$

Solution

$(i)\ \int x\operatorname{sec}h^2 x^2\ dx = \dfrac{1}{2}\int \operatorname{sec}h^2 x^2\ (2x\,dx) = \dfrac{1}{2}\tanh x^2 + c.$

$(ii)\ \int \tanh 3x\ dx = \int \dfrac{\sinh 3x}{\cosh 3x}\,dx = \dfrac{1}{3}\int \dfrac{3\sinh 3x}{\cosh 3x}\,dx$

$= \dfrac{1}{3}\ln(\cosh 3x) + c\ .$

$(iii)\ \int \sinh^2 x\ dx = \int \dfrac{\cosh 2x - 1}{2}\,dx = \dfrac{1}{2}\left[\dfrac{\sinh 2x}{2} - x\right] + c.$

$(iv)\ \int 4e^x \sinh x\ dx = \int 4e^x\,\dfrac{e^x - e^{-x}}{2}\,dx = \int (2e^{2x} - 2)\,dx = e^{2x} - 2x + c.$

Exercises (3.3)

(I) Approximate the following to four decimal places:

(1) $\cosh(\ln 4)$ (2) $\sinh(3)$ (3) $\tanh(-3)$ (4) $\text{sec}h(2.3)$ (5) $\coth(10)$

(6) $\tanh(-2)$ (7) $\text{csc}h(-1)$ (8) $\cosh(2.1)$ (9) $\coth(6)$ (10) $\text{sec}h(3.2)$

(II) Find $\frac{dy}{dx}$ for the following functions,

(1) $y = \sinh(5x)$ (2) $y = \sinh(3x-1)$ (3) $y = \sqrt{x}\,\tanh\sqrt{x}$

(4) $y = x^3\,\tanh x^2$ (5) $y = \cosh\left(\frac{1}{x}\right)$ (6) $y = \text{sec}h^3\,x^4$

(7) $y = \ln\sinh(2x)$ (8) $y = \cosh(\ln x)$ (9) $y = \coth(\ln x)$

(10) $y = \ln(\coth x)$ (11) $y = \tan^{-1}(\text{csc}h\,x)$ (12) $y = \sin^{-1}(\text{sec}h\,x)$

(III) Evaluate the following integrals.

(1) $\int x^2\cosh x^3\,dx$ (2) $\int \text{sec}h^2(2x-1)\,dx$ (3) $\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$

(4) $\int x^2\cosh x^3\,dx$ (5) $\int \frac{1}{\cosh^2 3x}\,dx$ (6) $\int \frac{1}{\sinh^2 2x}\,dx$

(7) $\int \tanh 3x\,\text{sec}h\,3x\,dx$ (8) $\int \coth 2x\,\text{csc}h\,2x\,dx$ (9) $\int \cosh x\,\text{csc}h^2\,x\,dx$

(10) $\int \frac{\cosh 2x}{1+\sinh 2x}\,dx$ (11) $\int \coth x\,dx$ (12) $\int e^{3x}\cosh e^{3x}\,dx$

3.4 Inverse Hyperbolic Functions

It is clear from the graphs of the hyperbolic functions that $\sinh x$, $\tanh x$, $\coth x$ and $\text{csc}h\,x$ are one-to-one for all values of x, while the graphs of $\cosh x$, and $\text{sec}h\,x$ are one-to-one for $x \geq 0$ only, so they are invertible. The graph of the inverse hyperbolic functions were obtained by reflecting the graphs of the hyperbolic functions about the line $y = x$.

3.4.1 Basic Definitions

Since the hyperbolic functions are expressible in terms of e^x, then the inverse hyperbolic functions are expressible in terms of $\ln x$ as in the following theorem.

Theorem (3.4.1)

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ $\operatorname{sec}h^{-1} x = \ln\left(\frac{1+\sqrt{1 - x^2}}{x}\right)$

Proof

(i) Let $y = \sinh^{-1} x,$ then $x = \sinh y = \frac{e^y - e^{-y}}{2}$, which can be rewritten as,

$e^y - 2x - e^{-y} = 0$. Multiplying both sides by e^y,

$e^{2y} - 2xe^y - 1 = 0$. Solve by quadratic formula,

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1} .$$

Since $e^y > 0$, refuse the minus sign.

Thus $e^y = x + \sqrt{x^2 + 1}$. Taking the natural logarithms yields,

$y = \ln\left(x + \sqrt{x^2 + 1}\right)$, *or* $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Similarly, we can prove the formula for $\cosh^{-1} x$ and $\operatorname{sec}h^{-1}x$.

(iii) Let $y = \tanh^{-1} x,$ then $x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$, which can be rewritten as,

$(x - 1)\, e^y = -(x + 1)\, e^{-y}$. Multiplying both sides by e^y,

$e^{2y} = \left(\frac{1+x}{1-x}\right)$. Solve by quadratic formula, and take the natural logarithms, we obtain,

$y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$, *or* $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$.

3.4.2 Derivatives and Integrals

The derivative of the inverse hyperbolic functions may be obtained from direct differentiating the logarithmic expressions above or similar to that done in inverse trigonometric functions.

$$\frac{d}{dx}\sinh^{-1}x = \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{x+\sqrt{x^2+1}}{(x+\sqrt{x^2+1})\,(\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}\,.$$

In another way, let $y=\sinh^{-1}x$, then $\sinh y = x$.

Differentiate both sides w.r.t. *x*, $\cosh y\,\frac{dy}{dx} = 1$,

$$\frac{d}{dx}\sinh^{-1}x = \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 x+1}} = \frac{1}{\sqrt{x^2+1}}\,.$$

In the same way and if $u=f(x)$ is differentiable , then by using chain rule we can show the following relations,

Theorem (3.4.2)

$$\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}\,, \qquad \frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}\,,\; u>1$$

$$\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}\,,\; |u|<1\,, \qquad \frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}\,,\; 0<u<1$$

Example (1) Find dy/dx for the following functions,

$(i)\; y = \sinh^{-1}x^3$ $\qquad (ii)\; y = \operatorname{sech}^{-1}x^2$

$(iii)\; y = \cosh^{-1}\left(e^{2x}\right)$ $\qquad (iv)\; y = \tanh^{-1}(\ln x)$

Solution

$$(i)\; \frac{dy}{dx} = \frac{1}{\sqrt{x^6+1}}\frac{d}{dx}x^3 = \frac{3x^2}{\sqrt{x^6+1}}.$$

$$(ii)\; \frac{dy}{dx} = \frac{-1}{x^2\sqrt{1-x^4}}\frac{d}{dx}x^2 = \frac{-2}{x\sqrt{1-x^4}}$$

$$(iii)\; \frac{dy}{dx} = \frac{1}{\sqrt{e^{4x}-1}}\frac{d}{dx}e^{2x} = \frac{2e^{2x}}{\sqrt{e^{4x}-1}}$$

$(iv)\ \dfrac{dy}{dx} = \dfrac{1}{1-(\ln x)^2}\dfrac{d}{dx}\ln x = \dfrac{1}{x\left[1-(\ln x)^2\right]}.$

We may use theorem (3.4.2) for differentiation to obtain the following integration formulas

<u>Theorem (3.4.3)</u>

(i) $\displaystyle\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$ (ii) $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$

(iii) $\displaystyle\int \frac{du}{1-u^2} = \tanh^{-1} u + c$ (iv) $\displaystyle\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sec}h^{-1} u + c$

These formulas can be generalized for $a > 0$ as follows ,

(i) $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c$ (ii) $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + c$

(iii) $\displaystyle\int \frac{du}{a^2-u^2} = \left(\frac{1}{a}\right)\tanh^{-1}\left(\frac{u}{a}\right)+c$ (iv) $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\left(\frac{1}{a}\right)\operatorname{sec}h^{-1}\left(\frac{u}{a}\right)+c$

Example (2) Evaluate the following integrals,

$(i)\ \displaystyle\int \frac{dx}{\sqrt{1+9x^2}}$ $(ii)\ \displaystyle\int \frac{e^{2x}\,dx}{25-e^{4x}}$

$(iii)\ \displaystyle\int \frac{dx}{\sqrt{4x^2-9}}$ $(iv)\ \displaystyle\int \frac{dx}{x\sqrt{16-x^4}}$

Solution

$$(i)\ \int \frac{dx}{\sqrt{1+9x^2}} = \frac{1}{3}\int \frac{3\,dx}{\sqrt{1+(3x)^2}} = \frac{1}{3}\sinh^{-1}(3x) + c$$

$$(ii)\ \int \frac{e^{2x}\,dx}{25-e^{4x}} = \frac{1}{2}\int \frac{2e^{2x}\,dx}{25-e^{4x}} = \left(\frac{1}{10}\right)\tanh^{-1} e^{2x} + c.$$

$$(iii)\ \int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{4x^2-9}} = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + c.$$

$(iv)\int \frac{dx}{x\sqrt{16-x^4}} = \frac{1}{2}\int \frac{2x\,dx}{x^2\sqrt{16-x^4}} = \left(\frac{-1}{8}\right)\operatorname{sec}h^{-1}\left(\frac{x^2}{4}\right) + c.$

Example (3) Evaluate the following integrals,

$(i)\ \int \frac{x\,dx}{\sqrt{x^4-25}}$ $(ii)\ \int \frac{dx}{\sqrt{x}\,(1-x)}$

$(iii)\ \int \frac{dx}{x\sqrt{4-x^8}}$ $(iv)\ \int \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Solution

$$(i)\int \frac{x\,dx}{\sqrt{x^4-25}} = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^4-25}} = \frac{1}{2}\cosh^{-1}\left(\frac{x^2}{5}\right) + c.$$

$$(ii)\int \frac{dx}{\sqrt{x}\,(1-x)} = 2\int \frac{\left(\frac{1}{2\sqrt{x}}\right)dx}{1-\left(\sqrt{x}\right)^2} = 2\tanh^{-1}\left(\sqrt{x}\right) + c.$$

$$(iii)\int \frac{dx}{x\sqrt{1-x^8}} = \frac{1}{4}\int \frac{4x^3\,dx}{x^4\sqrt{1-x^8}} = \frac{-1}{4}\operatorname{sec}h^{-1} x^4 + c.$$

$$(iv)\int \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int \frac{\left(\frac{1}{x}\right)dx}{\sqrt{1+(\ln x)^2}} = \sinh^{-1}(\ln x) + c.$$

Exercises (3.4)

(I) Find $\frac{dy}{dx}$ for the following functions,

(1) $y = \sinh^{-1}(5x)$ (2) $y = \sinh^{-1}(e^x)$ (3) $y = \cosh^{-1}\sqrt{x}$ (4) $y = \cosh^{-1}(\ln x)$

(5) $y = \tanh^{-1}(-4x)$ (6) $y = \tanh^{-1}(x^4)$ (7) $y = \operatorname{sec}h^{-1} x^2$ (8) $y = e^{\tanh^{-1} x^3}$

(9) $y = x\sinh^{-1}\frac{1}{x}$. (10) $y = \operatorname{sec}h^{-1}\sqrt{x}$ (11) $y = \ln(\cosh^{-1} 4x)$ (12) $y = \ln(\sinh^{-1} x)$

(13) $y = \tanh^{-1}(x+1)$. (14) $y = e^x \operatorname{sec}h^{-1}\sqrt{x}$ (15) $y = \cosh^{-1}(\sinh^{-1} x)$

(II) Evaluate the following integrals,

(1) $\int \frac{1}{\sqrt{81+16x^2}}\,dx$ (2) $\int \frac{x\,dx}{\sqrt{1+9x^4}}$ (3) $\int \frac{1}{49-4x^2}\,dx$

(4) $\int \frac{e^x}{\sqrt{e^{2x}+9}}\,dx$ (5) $\int \frac{e^x}{\sqrt{e^{2x}-16}}\,dx$ (6) $\int \frac{\sinh^{-1} x}{\sqrt{1-x^2}}\,dx$

(7) $\int \frac{1}{x\sqrt{9-x^4}}\,dx$ (8) $\int \frac{x\,dx}{\sqrt{4+x^4}}$ (9) $\int \frac{1}{4-9x^2}\,dx$

(10) $\int \frac{dx}{x\left(25-[\ln x]^2\right)}$ (11) $\int \frac{1}{\sqrt{9+4x^2}}\,dx$ (12) $\int \frac{dx}{x\sqrt{1+[\ln x]^2}}$

Chapter (4)

Techniques of Integration

4.1 Basic Integration Formulas

Now, we can summarize the basic rules derived in chapters (1) and (2), to be the base for different techniques of integration. The following table shows these formulas, where $u = f(x)$

(1) $\int u^n\, du = \begin{cases} \dfrac{u^{n+1}}{n+1} + c; & if\ n \neq -1 \\ \ln\lvert u\rvert + c; & if\ n = -1. \end{cases}$	
(2) $\int \dfrac{du}{u} = \ln\lvert u\rvert + c.$	(3) $\int e^u\, du = e^{u)} + c.$
(4) $\int a^u\, du = \dfrac{a^{u)}}{\ln a} + c.$	(5) $\log_a u = \dfrac{\ln n}{\ln a}$
(6) $\int \sin u\, du = -\cos u + c.$	(7) $\int \cos u\, du = \sin u + c.$
(8) $\int \sec^2 u\, du = \tan u + c.$	(9) $\int \csc^2 u\, du = -\cot u + c.$
(10) $\int \sec u \tan u\, du = \sec u + c.$	(11) $\int \csc u \cot u\, du = -\csc u + c.$
(12) $\int \tan u\, du = -\ln\lvert \cos u\rvert + c = \ln\lvert \sec u\rvert + c.$	(13) $\int \cot u\, du = \ln\lvert \sin u\rvert + c.$
(14) $\int \sec u\, du = \ln\lvert \sec u + \tan u\rvert + c.$	(15) $\int \csc u\, du = \ln\lvert \csc u - \cot u\rvert + c.$
(16) $\int \sinh u\, du = \cosh u + c$	(17) $\int \cosh u\, du = \sinh u + c.$
(18) $\int \sec h^2 u\, du = \tanh u + c$	(19) $\int \csc h^2 u\, du = -\coth u + c.$
(20) $\int \sec hu \tanh u\, du = -\sec hu + c$	(21) $\int \csc hu \coth u\, du = -\csc hu + c.$
(22) $\int \tanh u\, du = \ln\lvert \cosh u\rvert + c$	(23) $\int \coth u\, du = \ln\lvert \sinh u\rvert + c.$
(24) $\int \csc hu\, du = \ln\lvert \csc hu - \coth u\rvert + c$	(25) $\int \sec hu\, du = \tan^{-1} e^u + c.$
(26) $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + c$	(27) $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + c.$

$$(28)\ \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + c \qquad (29)\ \int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c.$$

$$(30)\ \int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + c \qquad (31)\ \int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + c$$

$$(32)\ \int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{-1}{a}\operatorname{sec}h^{-1}\left(\frac{u}{a}\right) + c \qquad (33)\ \int \frac{du}{u\sqrt{a^2 + u^2}} = \frac{-1}{a}\operatorname{csc}h^{-1}\left(\frac{u}{a}\right) + c$$

4.2. Integration by Parts

The familiar differential formula for the product,

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}.$$

In its differential form, the rule become,

$$d\,(u\,v) = u\,dv + v\,du.$$

Integration of both sides gives,

$$\int u\,dv = u\,v - \int v\,du.$$

This form is called **integration by parts**, which is a technique for simplifying integrals of the form,

$$\int f(x)\,g(x)\,dx$$

which not treated by the above basic integral forms.

Now with a proper choice of u and v the integral on the left hand side is obtained in terms of the integral on the right hand side, whenever that on the right is easier to be evaluated than the original one. The integration by parts formula may be used more than once in the same problem.

The form of the integral has many cases.

Case (1) : One of the two functions is polynomial

In this case, choose the polynomial to be differentiated, i.e. $u = x^n$, and dv is Integra bile, and note that we will use the integration by parts formula *n-times.*

Example (1) Evaluate, $\int x\, e^x\, dx$.

Solution

Let $\quad u = x, \quad dv = e^x\, dx\,, \quad ,$

then $\quad du = dx\,, \quad v = e^x$.

$$\int x\, e^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + c.$$

Example (2) Evaluate, $\int x \cos x\, dx$.

Solution

Let $\quad u = x, \quad dv = \cos x\, dx,$

then $\quad du = dx\,, \quad v = \sin x$.

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + c.$$

Example (3) Evaluate, $\int_0^{\pi/3} x \sec^2 x\, dx$.

Solution

Let $u = x$, $dv = \sec^2 x\, dx$. then, $du = dx$, $v = \tan x$.

$$\int_0^{\pi/3} x \sec^2 x\, dx = x \tan x \Big]_0^{\pi/3} - \int_0^{\pi/3} \tan x\, dx = x \tan x + ln\left|\cos x\right|\Big]_0^{\pi/3}$$

$$= \frac{\pi}{3} \tan\left(\frac{\pi}{3}\right) + \ln\left|\cos\left(\frac{\pi}{3}\right)\right| - \ln\left|\cos(0)\right| = \frac{\pi}{3}\sqrt{3} + \ln\left(\frac{1}{2}\right) = 1.12$$

Example (4) Evaluate, $\int x \ln x\, dx$.

Solution In this example, if we choose $u = x$ and $dv = \ln x\, dx$, then dv is not integrable, so we choose

$$u = \ln x, \quad dv = x\, dx\,, \text{ then } \quad du = \frac{1}{x} dx\,, \quad v = \frac{x^2}{2}.$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} . \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c.$$

Example (5) Evaluate, $\int x^2 \ e^{2x} \ dx$.

Solution

Let $u = x^2$ $dv = e^{2x} \, dx$. Then $du = 2x \, dx$, $v = 2e^{2x}$.

$$\int x^2 \ e^{2x} \ dx = 2x^2 e^{2x} - 4\int x e^{2x} \ dx \ .$$

We use integration by parts again, let $u = x$ and $dv = e^{2x} \, dx$, then $du = dx$, $v = 2e^{2x}$. and

$$\int x^2 \ e^{2x} \ dx = 2x^2 e^{2x} - 4[2x e^{2x} - 2\int e^{2x} \ dx] \ .$$

$$= 2x^2 e^{2x} - 8x e^{2x} + 4 \, e^{2x} + c.$$

Example (6) Evaluate, $\int x^3 \cos x^2 \ dx$.

Solution

$$\int x^3 \cos x^2 \ dx = \int x^2 \ (x \cos x^2) \ dx$$

Let $u = x^2$, $dv = x \cos x^2 \ dx$, then $du = 2x \, dx$, $v = \frac{1}{2} \sin x^2$.

$$\int x^3 \cos x^2 \ dx = \frac{1}{2} x^2 \sin x^2 - \int x \sin x^2 \ dx$$

$$= \frac{1}{2} x^2 \sin x^2 + \frac{1}{2} \cos x^2 + c.$$

<u>Case (2): The integral not contains polynomial</u>

Choose the easier function for integration to be integrated,

Example (7) Evaluate, $\int e^x \cos x \ dx$.

Solution

Let $u = e^x$ $dv = \cos x \ dx$, then $du = e^x \ dx$, $v = \sin x$.

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \ dx .$$

Again, use integration by parts, let $u = e^x$, $dv = \sin x \; dx$, then $du = e^x \; dx$, $v = -\cos x$ and,

$$\int e^x \cos x \, dx = e^x \sin x - [-e^x \cos x + \int e^x \cos x \, dx].$$

The unknown integral now appears in both sides of the equation. Combining the two expressions gives,

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + c.$$

Thus,

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x + c\right).$$

Example (8) Evaluate, $\int \sec^3 x \; dx$.

Solution

$$\int \sec^3 x \; dx = \int \sec x \; \sec^2 x \; dx$$

Let $u = \sec x$, $dv = \sec^2 x \; dx$, then $du = \sec x \tan x \, dx$, $v = \tan x$.

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \,(\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

Now, the unknown integral appears in both sides of the equation,

$$2\int \sec^3 x \; dx = \sec x \tan x + \ln\left|\sec x + \tan x\right| + c.$$

$$\int \sec^3 x \; dx = \frac{1}{2}\left(\sec x \tan x + \ln\left|\sec x + \tan x\right| + c\right).$$

<u>**Case (3)**</u> The integral contain one function not treated by one of the basic rules. In This case we take this function to be differentiated and integrate *dx*.

Example (9) Evaluate, $\int \ln x \; dx$.

Solution

Let $u = \ln x$, $dv = dx$, then $du = 1/x \, dx$, $v = x$.

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x}\, x \, dx = x \ln x - x + c.$$

Example (10) Evaluate, $\int \sin^{-1} x \; dx$.

Solution

Let $u = \sin^{-1} x$, $dv = dx$., then $du = \dfrac{1}{\sqrt{1-x^2}}\, dx$, $v = x$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + c.$$

Example (11) Evaluate, $\int \tanh^{-1} x \; dx$.

Solution

Let $u = \tanh^{-1} x$, $dv = dx$. Then $du = \dfrac{1}{1-x^2} dx$, $v = x$

$$\int \tanh^{-1} x \, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2} \, dx$$

$$= x \tanh^{-1} x + \frac{1}{2} \ln\left|1-x^2\right| + c.$$

So we can calculate the integral of all inverse trigonometric and inverse hyperbolic functions.

Case (4). Reduction Formulas for Integrals

Integration by parts may sometimes be employed to obtain reduction formulas for integrals. We can use such formulas to write an integral involving powers of an expression in terms of integrals that involve lower powers of the expression.

Example (12) Find a reduction formula for $\int \sin^n x \, dx$.

Solution $\int \sin^n x \, dx = \sin^{n-1} x \; \sin x \, dx$.

Let $u = \sin^{n-1} x$, and $dv = \sin x \, dx$, then

$du = (n-1) \sin^{n-2} x \cos x \, dx$, $v = -\cos x$. So

$$\int \sin^n x\, dx = \sin^{n-1} x \sin x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$n\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx$$

$$\boxed{\int \sin^n x\, dx = \left(\frac{-1}{n}\right)\sin^{n-1} x \cos x + \left(\frac{n-1}{n}\right)\int \sin^{n-2} x\, dx\ .}$$

Example (13) Use the reduction formula for $\int \sin^n x\, dx$ to evaluate

(i) $\int \sin^4 x\, dx$. (ii) $\int \sin^5 x\, dx$.

Solution

(i) Use the reduction formula for $n = 4$,

$$\int \sin^4 x\, dx = \left(\frac{-1}{4}\right)\sin^3 x \cos x + \left(\frac{3}{4}\right)\int \sin^2 x\, dx$$

Again, use the reduction formula for $n = 2$,

$$\int \sin^4 x\, dx = \left(\frac{-1}{4}\right)\sin^3 x \cos x + \left(\frac{3}{4}\right)\left(\left(\frac{-1}{2}\right)\sin x \cos x + \left(\frac{1}{2}\right)\int dx\right)$$

$$= \left(\frac{-1}{4}\right)\sin^3 x \cos x + \left(\frac{-3}{8}\right)\sin x \cos x + \left(\frac{3}{8}\right)x + c.$$

(ii) Use the reduction formula for $n = 5$,

$$\int \sin^5 x\, dx = \left(\frac{-1}{5}\right)\sin^4 x \cos x + \left(\frac{4}{5}\right)\int \sin^3 x\, dx$$

Again, use the reduction formula for $n = 3$,

$$\int \sin^5 x\, dx = \frac{-\sin^3 x \cos x}{5} + \left(\frac{4}{5}\right)\left(\left(\frac{-1}{3}\right)\sin^2 x \cos x + \left(\frac{2}{3}\right)\int \sin x\, dx\right)$$

$$= \frac{-\sin^3 x \cos x}{5} - \frac{4\sin^2 x \cos x}{15} - \frac{8\cos x}{15} + c.$$

Tabular Integration

In case (1) where one of the two functions can be differentiated repeatedly to become zero, and if many repetitions are required, the calculations can be cumbersome. In situation like this, there is a way to organize the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

Example (14) Evaluate $\int x^2 \cos x \, dx$.

Solution Let $f(x) = x^2$ and $g(x) = \cos x$, we list,

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	+	$\cos x$
2x	-	$\sin x$
2	+	$-\cos x$
0		$-\sin x$

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x\cos x - 2\sin x + C.$$

Example (15) Evaluate $\int x^3 e^{2x} \, dx$.

Solution Let $f(x) = x^3$ and $g(x) = e^{2x}$, we list,

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	+	e^{2x}
$3x^2$	-	$e^{2x}/2$
$6x$	+	$e^{2x}/4$
6	-	$e^{2x}/8$
0		$e^{2x}/16$

$$\int x^3 e^{2x} \, dx = x^3 \frac{e^{2x}}{2} - 3x^2 \frac{e^{2x}}{4} + 6x \frac{e^{2x}}{8} - 6 \frac{e^{2x}}{16} + c$$

$$= \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x} + c .$$

Exercises (4.2)

(I) Evaluate the following integral,

(1) $\int x\, e^x \, dx$ (2) $\int x\, e^{2x} \, dx$ (3) $\int x^2\, e^{3x} \, dx$

(4) $\int x \sin x \, dx$ (5) $\int x \cos 5x \, dx$ (6) $\int x \sin 3x \, dx$

(7) $\int x \sec x \tan x \, dx$ (8) $\int x^2 \cos x \, dx$ (9) $\int \tan^{-1} x \, dx$

(10) $\int e^{2x} \sin 2x \, dx$ (11) $\int \sqrt{x} \ln x \, dx$ (12) $\int \cos^{-1} x \, dx$

(13) $\int e^{-x} \sin x \, dx$ (14) $\int \tanh^{-1} x \, dx$ (15) $\int \sin x \ln \cos x \, dx$

(16) $\int x^3 \ln x \, dx$ (17) $\int \csc^3 x \, dx$ (18) $\int \sec h^{-1} x \, dx$

(19) $\int_0^{\pi/2} x \sin 2x \, dx$ (20) $\int x \tan^2 x \, dx$ (21) $\int_0^1 \frac{x^3}{\sqrt{x^2+1}} \, dx$

(II) Use integration by parts to derive the reduction formula

(1) $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$

(2) $\int \tan^n x \, dx = \left(\frac{1}{n-1}\right) \tan^{n-1} x - \int \tan^{n-2} x \, dx$.

(3) $\int (\ln x)^n \, dx = x (\ln x)^n - n \int (\ln x)^{n-1} \, dx$

(4) $\int \cos^n x \, dx = \left(\frac{1}{n}\right) \cos^{n-1} x \sin x + \left(\frac{n-1}{n}\right) \int \cos^{n-2} x \, dx$.

(5) $\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

(III) Use Exercise (II) to evaluate

(1) $\int x^5 e^x \, dx$ (2) $\int x^4 e^x \, dx$ (3) $\int (\ln x)^4 \, dx$

(4) $\int (\ln x)^5 \, dx$ (5) $\int \sec^5 x \, dx$ (6) $\int \sec^4 x \, dx$

(7) $\int \cos^5 x \, dx$ (8) $\int \cos^4 x \, dx$ (9) $\int \tan^3 x \, dx$

4.3 Trigonometric Integrals

In the previous section we obtained a reduction formula for the integrals of $\sin^n x$ and $\cos^n x$ but we can do this integrals without the reduction formulas as follows.

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

If the power more than 2, we can use the general cases

(I) $$\int \sin^m x \cos^n x\, dx.$$

We will discuss the integral for different values of *m, n*.

If *m* is an odd integer.

The integral takes the form,

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cos^n x \sin x\, dx$$

Use the identity $\sin^2 x = 1 - \cos^2 x$ to express the expression $\sin^{m-1} x \cos^n x$ in terms of $\cos x$, and integrate.

If *n* is an odd integer

The integral takes the form,

$$\int \sin^m x \cos^n x\, dx = \int \sin^m x \cos^{n-1} x \cos x\, dx$$

Use the identity $\cos^2 x = 1 - \sin^2 x$ to express the expression $\sin^m x \cos^{n-1} x$ in terms of $\sin x$, and integrate.

If *m and n* are even integers

Use the half-angle formulas: $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ and $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ to reduce the exponents.

Example (1) Evaluate : $\int \sin^3 x \cos^4 x\, dx$

Solution

$$\int \sin^3 x \cos^4 x \, dx = \int \sin^2 x \cos^4 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^4 x \sin x \, dx = \int (\cos^4 x - \cos^6 x) \sin x \, dx$$

$$= -\int (\cos^4 x - \cos^6 x)(-\sin x) \, dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c.$$

Example (2) Evaluate : $\int \sin^2 x \cos^5 x \, dx$

Solution

$$\int \sin^2 x \cos^5 x \, dx = \int \sin^2 x \cos^4 x \cos x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int \sin^2 x (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

$$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) \cos x \, dx$$

$$= \frac{\sin^3 x}{3} - \frac{2 \sin^5 x}{5} + \frac{\sin^7 x}{7} + c.$$

Example (3) Evaluate : $\int \cos^5 x \, dx$

Solution

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

$$= \sin x - \frac{2 \sin^3 x}{3} + \frac{\sin^5 x}{5} + c.$$

Example (4) Evaluate : $\int \sin^3 x \, dx$

Solution

$$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

$$= -\int (1 - \cos^2 x)(-\sin x) \, dx = -\cos x + \frac{\cos^3 x}{3} + c.$$

Example (5) Evaluate : $\int \sin^2 x \; dx$

Solution

$$\int \sin^2 x \; dx = \int \frac{1-\cos 2x}{2} \; dx = \frac{1}{2}\int (1-\cos 2x)\; dx$$

$$= \frac{1}{2}\,(x - \frac{\sin 2x}{2}) + c = \frac{x}{2} - \frac{\sin 2x}{4} + c.$$

Example (6) Evaluate : $\int \cos^2 x \; dx$

Solution

$$\int \cos^2 x \; dx = \int \frac{1+\cos 2x}{2} \; dx = \frac{1}{2}\int (1+\cos 2x)\; dx$$

$$= \frac{1}{2}\,(x + \frac{\sin 2x}{2}) + c = \frac{x}{2} + \frac{\sin 2x}{4} + c.$$

Example (7) Evaluate : $\int \sin^2 x \; \cos^2 x \, dx$

Solution

$$\int \sin^2 x \; \cos^2 x \, dx = \int \frac{1-\cos 2x}{2} \; \frac{1+\cos 2x}{2} dx = \frac{1}{4}\int (1-\cos^2 2x)\; dx$$

$$= \frac{1}{4}\int \left[1 - \left(\frac{1+\cos 4x}{2} \right) \right] dx = \frac{1}{8}\int (1-\cos 4x)\; dx$$

$$= \frac{1}{8}\,(x - \frac{\sin 4x}{4}) + c = \frac{x}{8} - \frac{\sin 4x}{32} + c.$$

Example (8) Evaluate : $\int \sin^4 x \; dx$

Solution

$$\int \sin^4 x \; dx = \int \left(\frac{1-\cos 2x}{2} \right)^2 \; dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\; dx$$

$$= \frac{1}{4}\int \left[1 - 2\cos 2x + \left(\frac{1+\cos 4x}{2} \right) \right] dx$$

$$= \frac{1}{4}\int \left[\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2} \right] dx$$

$$= \left(\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \right) + c.$$

Similar to the above form (I), we can treat the integral,

(II) $$\boxed{\int \tan^m x \ \sec^n x \ dx}$$

<u>If m is an odd integer</u>

The integral takes the form

$$\int \tan^m x \ \sec^n x \ dx = \int \tan^{m-1} x \ \sec^{n-1} x \ (\sec x \tan x) \ dx$$

Use the identity $\tan^2 x = \sec^2 x - 1$ to express the expression $\tan^{m-1} x \ \sec^{n-1} x$ in terms of $\sec x$, and integrate.

<u>If n is an even integer</u>

The integral takes the form,

$$\int \tan^m x \ \sec^n x \ dx = \int \tan^m x \ \sec^{n-2} x \ (\sec^2 x) \ dx$$

Use the identity $\tan^2 x = \sec^2 x - 1$ to express the expression $\tan^m x \ \sec^{n-2} x$ in terms of $\tan x$, and integrate.

<u>If m is even and n is an odd</u>

There is no standard method of evaluation. We may use integration by parts.

Example (6) Evaluate : $\int \tan^3 x \ \sec^5 x \ dx$

Solution

$$\int \tan^3 x \ \sec^5 x \ dx = \int \tan^2 x \ \sec^4 x \ (\sec x \tan x) \ dx$$

$$= \int (\sec^2 x - 1) \sec^4 x \ (\sec x \tan x) \ dx$$

$$= \int (\sec^6 x - \sec^4 x) \ (\sec x \tan x) \ dx$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + c.$$

Example (7) Evaluate : $\int \tan^2 x \sec^4 x \, dx$

Solution

$$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^2 x \, (\tan^2 x + 1) \sec^2 x \, dx$$

$$= \int (\tan^4 x + \tan^2 x) \sec^2 x \, dx$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + c$$

Integrals of the form, $\int \cot^m x \csc^n x \, dx$ may be evaluated in a similar method.

(III) $\int \sin mx \sin nx \, dx$, $\int \sin mx \cos nx \, dx$, $\int \cos mx \cos nx \, dx$

Here we use the product to sum formulas:

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

Example (8) Evaluate : $\int \cos 5x \sin 3x \, dx$

Solution

$$\int \cos 5x \sin 3x dx = \frac{1}{2}\int (\sin 8x + \sin(-2x))dx = \frac{-1}{16}\cos 8x - \frac{1}{4}\cos 2x + c.$$

Example (9) Evaluate : $\int \sin 4x \sin 2x\, dx$

Solution

$$\int \sin 4x \sin 2x\, dx = \frac{1}{2}\int (\cos 2x - \cos 6x)\, dx = \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + c.$$

Exercises (4.3)

(I) Evaluate the following integrals,

(1) $\int \cos^3 x\, dx$ (2) $\int \cos^5 x \sin x\, dx$ (3) $\int \sin^6 x\, dx$

(4) $\int \sin^2 x \cos^5 x\, dx$ (5) $\int \tan^3 x \sec^4 x\, dx$ (6) $\int \tan^2 2x \sec^3 2x\, dx$

(7) $\int \tan^3 x \sec^3 x\, dx$ (8) $\int \tan^6 x \sec x\, dx$ (9) $\int \tan^6 x\, dx$

(10) $\int \sec^4 x\, dx$ (11) $\int \sqrt{\sin x}\, \cos^3 x\, dx$ (12) $\int \cos^2 x \sin^5 x\, dx$

(13) $\int \sin 5x \sin 3x\, dx$ (14) $\int \cos^{1/5} x \sin x\, dx$ (15) $\int (\tan x + \cot x)^2\, dx$

(16) $\int \sin ax \cos ax\, dx$ (17) $\int_0^{\pi/2} \sin 3x \cos 2x\, dx$ (18) $\int \cos 2x \sin 3x\, dx$

(19) $\int \csc^4 x \cot^4 x\, dx$ (20) $\int \cot^2 x \csc^2 x\, dx$ (21) $\int \frac{\sec^2 x}{(1+\tan x)^2}\, dx$

(II) Prove that if m and n are positive integers,

$$\int \sin mx \sin nx\, dx = \begin{cases} \dfrac{\sin (m-n)x}{2(m-n)} - \dfrac{\sin (m+n)x}{2(m+n)} + c & \text{if } m \neq n \\ \dfrac{x}{2} - \dfrac{\sin 2m}{4m} + c & \text{if } m = n \end{cases}$$

(III) Let m, n be distinct nonnegative. Prove that:

(1) $\int_0^{2\pi} \sin mx \cos nx\, dx = 0$ (2) $\int_0^{2\pi} \cos mx \cos nx\, dx = 0$ (3) $\int_0^{2\pi} \sin mx \sin nx\, dx = 0$

(IV) Use the reduction formula to show that,

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx.$$

Then use this formula to evaluate,

(1) $\int_0^{\pi/2} \sin^3 x \ dx$ (2) $\int_0^{\pi/2} \sin^4 x \ dx$

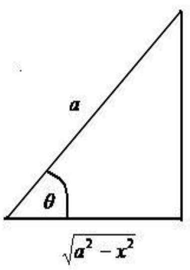
(3) $\int_0^{\pi/2} \sin^5 x \ dx$ (4) $\int_0^{\pi/2} \sin^6 x \ dx$

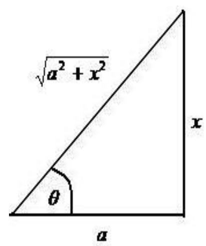
4.4 Trigonometric Substitutions

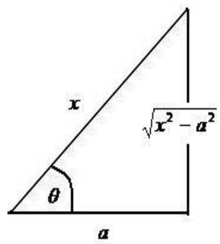
The technique here is useful for eliminating radicals from certain types of integrals. The substitutions are listed in the following table:

Expression in integrand	Trigonometric substitution	Element of integration
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$dx = a \cos \theta \ d\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$dx = a \sec^2 \theta \ d\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$dx = a \sec\theta \tan \theta \ d\theta$

We shall assume that θ is in the range of the corresponding inverse trigonometric function.







$x = a \sin\theta$ $x = a \tan\theta$ $x = a \sec\theta$

$\sqrt{a^2 - x^2} = a\cos\theta$ $\sqrt{a^2 + x^2} = a\sec\theta$ $\sqrt{x^2 - a^2} = a\tan\theta$

Example (1) Evaluate : $\int \frac{1}{x^2 \sqrt{16 - x^2}} \ dx$

Solution

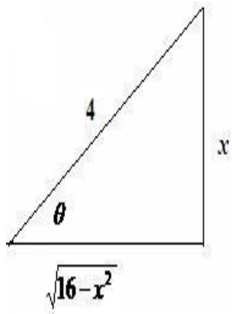
Let $x = 4\sin\theta$, then,

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = 4\cos\theta, \quad dx = 4\cos\theta\, d\theta$$

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = \int \frac{4\cos\theta\, d\theta}{(16\sin^2\theta)(4\cos\theta)} = \frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{16}\int \csc^2\theta\, d\theta = -\frac{1}{16}\cot\theta + c$$

We must now return to the original variable of integration, x. By drawing a right triangle that corresponding to $x = 4\sin\theta$, i.e. $\theta = \sin^{-1}(x/4)$.

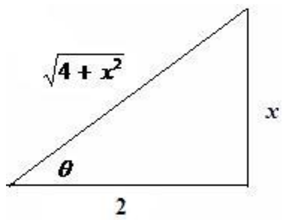


Then $\cot\theta = \dfrac{\sqrt{16-x^2}}{x}$.

Therefore,

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = -\frac{\sqrt{16-x^2}}{16x} + C.$$

Example (2) Evaluate : $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$



Solution

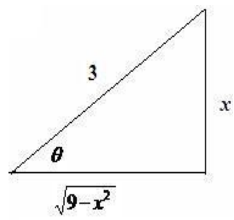
Let $x = 2\tan\theta$. Then, $dx = 2\sec^2\theta\, d\theta$, and,

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2\sec\theta$$

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + c = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

Example (3) Evaluate : $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$



Solution

Let $x = 3\sin\theta$, then,

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta, \quad dx = 3\cos\theta\, d\theta$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta\,\cos\theta\,d\theta}{3\cos\theta} = 9\int \sin^2\theta\,d\theta$$

$$= 9\int \frac{1-2\cos\theta}{2}\,d\theta = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + c$$

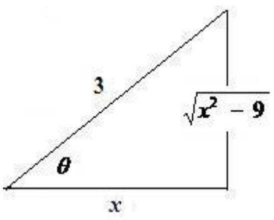
$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x\sqrt{9-x^2}}{9}\right) + c.$$

Example (4) Evaluate : $\int \frac{\sqrt{x^2-9}}{x}\,dx$

Solution

Let $x = 3\sec\theta$. then, $dx = 3\sec\theta\,\tan\theta\,d\theta$, and,

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\sqrt{\sec^2\theta - 1} = 3\tan\theta.$$

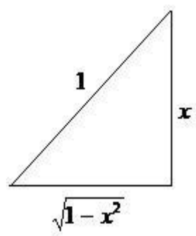


$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\,\tan\theta\,d\theta = 3\int \tan^2\theta\,d\theta$$

$$= 3\int (\sec^2\theta - 1)\,d\theta = 3\tan\theta - 3\theta + c.$$

$$= 3\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + c.$$

Example (5) Evaluate : $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$



Solution

Let $x = \sin\theta$, then, $dx = \cos\theta\,d\theta$

$$\int \frac{(1-x^2)^{3/2}}{x^6}\,dx = \int \frac{(1-\sin^2\theta)^{3/2}}{\sin^6\theta}\cos\theta\,d\theta = \int \frac{\cos^4\theta}{\sin^6\theta}\,d\theta$$

$$= \int \frac{\cos^4\theta}{\sin^4\theta}\frac{1}{\sin^2\theta}\,d\theta = \int \cot^4\theta\,\csc^2\theta\,d\theta$$

$$= -\int \cot^4\theta\,(-\csc^2\theta\,d\theta) = -\frac{\cot^5\theta}{5} + c = \frac{-1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + c$$

Exercises (4.4)

Evaluate the following integrals,

(1) $\int \frac{1}{x\sqrt{4-x^2}}\,dx$ (2) $\int \sqrt{4-x^2}\,dx$ (3) $\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$

(4) $\int \frac{x^2}{\sqrt{4+x^2}}\,dx$ (5) $\int \frac{1}{(x^2-1)^{3/2}}\,dx$ (6) $\int \frac{\sqrt{x^2-9}}{x}\,dx$

(7) $\int \frac{1}{(36+x^2)^2}\,dx$ (8) $\int \frac{1}{x^2\sqrt{4x^2-9}}\,dx$ (9) $\int \frac{x}{(16-x^2)^2}\,dx$

(10) $\int x^3\sqrt{16-x^2}\,dx$ (11) $\int \frac{x^3}{\sqrt{9x^2+49}}\,dx$ (12) $\int \frac{x^3}{(4+x^2)^{5/2}}\,dx$

(13) $\int \frac{1}{x^4\sqrt{x^2-3}}\,dx$ (14) $\int \frac{1}{x^4\sqrt{4+x^2}}\,dx$ (15) $\int \frac{(4+x^2)^2}{x^3}\,dx$

4.5 Integration of Rational Function Using Partial Fractions

A rational function is a ratio of two polynomials,

$$Q(x) = \frac{f(x)}{g(x)},$$

where $f(x)$ and $g(x)$ are polynomials.

If the degree of $f(x)$ is greater than or equal the degree of $g(x)$, use long division to obtain a function of the form,

$$Q(x) = k(x) + \frac{h(x)}{g(x)},$$

where the degree of $h(x)$ now is less than the degree of $g(x)$.

Now the rational faction , $\frac{h(x)}{g(x)}$ is a fraction, and no matter how complicated, it can be rewritten as a sum of simpler fractions that we can integrate with techniques we already know. This method of rewriting the complicated fraction as a sum of simpler fractions is called the **method of partial fractions.**

This method may be summarized as, express the polynomial $g(x)$ as a product of linear factor $(ax + b)$ or irreducible quadratic factors $(ax^2 + bx + c)$, and collect repeated factors so that $g(x)$ is a product of different factors of the form $(ax + b)^n$ and $(ax^2 + bx + c)^n$. Then apply the following two rules.

Rule (1). For each factor of the form $(ax + b)^n$, the partial fraction decomposition contains the following sum of n partial fractions:

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \ldots + \frac{A_n}{(ax + b)^n},$$

where $A_1, A_2, \ldots, A_n$ are constants to be determined. In the case where $n = 1$, only the first term in the sum appears.

Rule (2). For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition contains the following sum of n partial fractions:

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n},$$

where $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$, are constants to be determined. In the case where $n = 1$, only the first term in the sum appears.

Example (1) Evaluate : $\int \frac{5x - 10}{x^2 - 3x - 4}\, dx$

Solution

$$\frac{5x - 10}{x^2 - 3x - 4} = \frac{5x - 10}{(x - 4)(x + 1)} = \frac{A}{x - 4} + \frac{B}{x + 1} = \frac{A(x + 1) + B(x - 4)}{(x - 4)(x + 1)}$$

The two fractions in the left and right are the same, then

$$5x - 10 = A(x + 1) + B(x - 4)$$

The coefficients A *and* B may be determined by two methods

Method_1. Verify both sides for any value of x,

- For $x = -1,\ -15 = -5B,\ B = 3.$
- For $x = 4,\ 10 = 5A,\ A = 2.$

Method_2. Equating the coefficients on both sides for every powers of *x*,

$A + B = 5$, $A - 4B = -10$. Solve the two equations to find $A = 2$, $B = 3$. Then

$$\frac{5x-10}{x^2-3x-4} = \frac{2}{x-4} + \frac{3}{x+1}$$. Therefore

$$\int \frac{5x-10}{x^2-3x-4}\,dx = \int\left(\frac{2}{x-4} + \frac{3}{x+1}\right)dx$$

$$= 2\ln|x-4| + 3\ln|x+1| + c.$$

Example (2) Evaluate : $\int \frac{1}{x^2+x-2}\,dx$

Solution

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$A(x+2) + B(x-1) = 1$. Solve to find, $A = 1/3$, $B = -1/3$. Then,

$$\int \frac{1}{x^2+x-2}\,dx = \int \frac{1/3}{x-1} - \frac{1/3}{x+2}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c.$$

Example (3) Evaluate : $\int \frac{2x+4}{x^3-2x^2}\,dx$

Solution

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x+2)}$$

$Ax(x-2) + B(x-2) + Cx^2 = 2x + 4$. Solve to find

$A = B = -2$, $C = 2$. Then,

$$\int \frac{2x+4}{x^3-2x^2}\,dx = \int\left(\frac{-2}{x} - \frac{2}{x^2} + \frac{2}{x-2}\right)dx = -2\ln|x| + \frac{2}{x} + 2\ln|x-1| + c.$$

Example (4) Evaluate : $\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx$

Solution

$$\frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

$$= \frac{A(x-2)^3 + B(x+1)(x-2)^2 + C(x+1)(x-2) + D(x+1)}{(x+1)(x-2)^3}.$$

Then $3x^3 - 18x^2 + 29x - 4 = A(x-2)^3 + B(x+1)(x-2)^2 + C(x+1)(x-2) + D(x+1)$

Solve to find, $A = 2$, $B = 1$, $C = -3$, $D = 2$. Hence,

$$\int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3}\, dx = \int \left(\frac{2}{x+1} + \frac{1}{x-2} - \frac{3}{(x-2)^2} + \frac{2}{(x-2)^3} \right) dx$$

$$= 2\ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + c.$$

Example (5) Evaluate : $\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\, dx$

Solution

$$\frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} = \frac{x^2 + x - 2}{x^2(3x-1) + (3x-1)} = \frac{x^2 + x - 2}{(3x-1)(x^2+1)}$$

$$= \frac{A}{3x-1} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2+1) + (Bx + C)(3x-1)}{(3x-1)(x^2+1)}$$

$x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(3x - 1)$.

Solve to find: $A = -\frac{7}{5}$, $B = \frac{4}{5}$, $C = \frac{3}{5}$. Then,

$$\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\, dx = \int \frac{-7/5}{3x-1} + \frac{(4/5)x + (3/5)}{x^2 + 1}\, dx$$

$$= \frac{-7}{5} \int \frac{dx}{3x-1} + \frac{1}{5} \int \frac{4x + 3}{x^2 + 1}\, dx$$

$$= \frac{-7}{15} \int \frac{3dx}{3x-1} + \frac{2}{5} \int \frac{2x\,dx}{x^2 + 1} + \frac{3}{5} \int \frac{dx}{x^2 + 1}$$

$$= \frac{-7}{15} \ln|3x - 1| + \frac{2}{5} \ln|x^2 + 1| + \frac{3}{5} \tan^{-1} x + c.$$

Example (6). Evaluate : $\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\, dx$

Solution.

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2+3)^2}$$

$$= \frac{A(x^2+3)^2 + (Bx+C)\ (x^2+3)(x+2) + (Dx+E)\ (x+2)}{(x+2)\ (x^2+3)^2}$$

then

$$3x^4 + 4x^3 + 16x^2 + 20x + 9 =$$
$$= A(x^2+3)^2 + (Bx+C)\ (x^2+3)(x+2) + (Dx+E)\ (x+2)$$

Solve (more complicated) to find, $A = 1,\ B = 2,\ C = 0,\ D = 4, E = 0$. Then

$$\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)\ (x^2+3)^2}\, dx = = \int \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}\, dx$$

$$= \ln\left| x+2 \right| + \ln\left| x^2+3 \right| - \frac{2}{x^2+3} + c.$$

Example (7). Evaluate : $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx$

Solution.

In this example the numerator has degree 4 and the denominator has degree 2. Thus, we first perform a long division to obtain

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx = \int (3x^2+1) + \frac{1}{x^2+x-2}\, dx$$

The second integral was treated by partial fractions in example (2), hence,

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx = x^3 + x + \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c.$$

<u>Exercises (4.5)</u>

Evaluate the following integrals,

(1) $\int \frac{5x-12}{x(x-4)}\, dx$ (2) $\int \frac{dx}{x^2+3x-4}$

(3) $\int \frac{37-11x}{(x+1)(x-2)(x-3)}\, dx$ (4) $\int \frac{2x^2-9x-9}{x^3-9x}\, dx$

(5) $\int \frac{6x-11}{(x-1)^2}\, dx$ (6) $\int \frac{1}{x(x^2-1)}\, dx$

(7) $\int \frac{x+16}{x^2-2x-8}\, dx$ (8) $\int \frac{2x^5 - x^3 - 1}{x^3 - 4x}\, dx$

(9) $\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}\,dx$ (10) $\int \frac{x^5 + 2x^2 + 1}{x^3 - x}\,dx$

(11) $\int \frac{x^3 + 3x - 2}{x^2 - x}\,dx$ (12) $\int \frac{1}{x(x^2 - 1)}\,dx$

(13) $\int \frac{10x^2 + 9x + 1}{2x^3 + 3x^2 + x}\,dx$ (14) $\int \frac{x^2 + x - 16}{(x+1)(x-3)^2}\,dx$

(15) $\int \frac{9x^4 + 17x^3 + 3x^2 - 8x + 3}{x^5 + 3x^4}\,dx$ (16) $\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}\,dx$

4.6 Integrals Involving a Quadratic Expression

Integrals that involve a quadratic expression $ax^2 + bx + c$, where $a \neq 0$ and $b \neq 0$, can often be evaluated by first completing the square, then making an appropriate substitution. The following examples illustrate this idea.

Example (1) Evaluate : $\int \frac{x}{x^2 - 4x + 8}\,dx$

Solution Completing the square yields

$$x^2 - 4x + 8 = x^2 - 4x + 4 - 4 + 8$$

$$= x^2 - 4x + 4 + 4 = (x-2)^2 + 4.$$

Thus, use the substitution, $u = x - 2, \quad du = dx$ yields

$$\int \frac{x}{x^2 - 4x + 8}\,dx = \int \frac{x}{(x-2)^2 + 4}\,dx = \int \frac{u + 2}{u^2 + 4}\,du$$

$$= \int \frac{u}{u^2+4}\,du + \int \frac{2}{u^2+4}\,du = \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\ln(u^2 + 4) + 2\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{u}{2}\right) + c$$

$$= \frac{1}{2}\ln((x-2)^2 + 4) + \tan^{-1}\left(\frac{x-2}{2}\right) + c..$$

Example (2) Evaluate : $\int \frac{2x - 1}{x^2 - 6x + 13}\,dx$

Solution Completing the square yields

$$x^2 - 6x + 13 = x^2 - 6x + 9 - 9 + 13 = (x-3)^2 + 4.$$

Thus, use the substitution, $u = x - 3,\quad du = dx$ yields

$$\int \frac{2x-1}{x^2 - 6x + 13}\,dx = \int \frac{2x-1}{(x-3)^2 + 4}\,dx = \int \frac{2u + 5}{u^2 + 4}\,du$$

$$= \int \frac{2u}{u^2+4}\,du + \int \frac{5}{u^2+4}\,du = \ln(u^2 + 4) + 5\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{u}{2}\right) + c$$

$$= \frac{1}{2}\ln((x-3)^2 + 4) + \frac{5}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c.$$

Example (3) Evaluate : $\displaystyle\int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx$

Solution Completing the square yields

$$x^2 + 8x + 25 = x^2 + 8x + 16 - 16 + 25 = (x + 4)^2 + 9 .$$

Thus, use the substitution, $u = x + 4,\quad du = dx$ yields

$$\int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx = \int \frac{1}{\sqrt{(x+4)^2 + 9}}\,dx = \int \frac{1}{\sqrt{u^2 + 9}}\,du$$

$$= \sinh^{-1}\left(\frac{u}{3}\right) + c = \sinh^{-1}\left(\frac{x+4}{3}\right) + c .$$

Example (4) Evaluate : $\displaystyle\int \frac{1}{\sqrt{8 + 2x - x^2}}\,dx$

Solution Completing the square yields

$$8 + 2x - x^2 = 8 - (x^2 - 2x) = 8 + 1 - (x^2 - 2x + 1) = 9 - (x-1)^2$$

Thus, use the substitution, $u = x - 1,\quad du = dx$ yields

$$\int \frac{1}{\sqrt{8 + 2x - x^2}}\,dx = \int \frac{1}{\sqrt{9 - (x-1)^2}}\,dx = \int \frac{1}{\sqrt{9 - u^2}}\,du$$

$$= \sin^{-1}\left(\frac{u}{3}\right) + c = \sin^{-1}\left(\frac{x-1}{3}\right) + c .$$

Exercises (4.6)

Evaluate the following integrals,

(1) $\int \frac{1}{x^2 + 2x + 5}\, dx$ (2) $\int \frac{dx}{x^2 - 4x + 13}$ (3) $\int \frac{1}{x^2 - 4x + 8}\, dx$

(4) $\int \frac{x\, dx}{x^2 + 6x + 10}$ (5) $\int \frac{1}{\sqrt{4x - x^2}}\, dx$ (6) $\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$

(7) $\int \frac{1}{\sqrt{9 - 8x - x^2}}\, dx$ (8) $\int \sqrt{4 - x^2}\, dx$ (9) $\int \frac{1}{\sqrt{x^2 + 4x + 5}}\, dx$

(10) $\int x^3 \sqrt{16 - x^2}\, dx$ (11) $\int \frac{1}{\sqrt{x^2 + 6x + 13}}\, dx$ (12) $\int \frac{x^2}{\sqrt{4 + x^2}}\, dx$

(13) $\int \frac{1}{2x^2 - 3x + 9}\, dx$ (14) $\int \frac{x^2}{\sqrt{9 - x^2}}\, dx$ (15) $\int \frac{e^x}{e^{2x} - 3e^x + 2}\, dx$

(16) $\int e^x \sqrt{1 - e^{2x}}\, dx$ (17) $\int_2^3 \frac{x^2 - 4x + 6}{x^2 - 4x - 5}\, dx$ (18) $\int \frac{e^x\, dx}{\sqrt{1 + e^x + e^{2x}}}$

4.7 Miscellaneous Substitution

In this section we shall consider substitutions that are useful for evaluating certain types of integrals.

The following examples illustrate different substitutions.

Example (1) Evaluate, $\int \frac{x^3}{\sqrt[3]{x^2 + 4}}\, dx$.

Solution

The substitution $u = \sqrt[3]{x^2 + 4}$ leads to,

$u^3 = x^2 + 4$ or $x^2 = u^3 - 4$ and $2x\, dx = 3u^2\, du$, then,

$$\int \frac{x^3}{\sqrt[3]{x^2 + 4}}\, dx = \frac{1}{2}\int \frac{x^2}{\sqrt[3]{x^2 + 4}}\,(2x dx) = \frac{3}{2}\int \frac{u^3 - 4}{u}\, u^2\, du$$

$$= \frac{3}{2}\int (u^4 - 4u)\, du = \frac{3}{2}\left(\frac{u^5}{5} - 2u^2\right) + c$$

$$= \frac{3}{10} u^2 (u^3 - 10) + c = \frac{3}{10}(x^2 + 4)^{2/3}(x^2 - 6) + c\,.$$

Another Solution

We can use the substitution $u = x^2 + 4$ that will leads to, $x^2 = u - 4$ and $2x\,dx = du$, then,

$$\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx = \frac{1}{2}\int \frac{x^2}{\sqrt[3]{x^2+4}}\,(2xdx) = \frac{1}{2}\int \frac{u-4}{u^{1/3}}\,du$$

$$= \frac{1}{2}\int (u^{2/3} - 4u^{-1/3})\,du = \frac{1}{2}\left(\frac{3u^{5/3}}{5} - 6u^{2/3}\right) + c$$

$$= \frac{3}{10}u^{2/3}(u-10)+c = \frac{3}{10}(x^2+4)^{2/3}(x^2-6)+c\,.$$

Example (2) Evaluate, $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$.

Solution

We can use a substitution that will eliminate the two radicals, we use $u^6 = x$ or $u = x^{1/6}$, then $dx = 6u^5\,du$,

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int \frac{1}{u^3+u^2}\,6u^5\,du = 6\int \frac{u^3}{u+1}\,du$$

By long division, $\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}$.

Then,

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = 6\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)du$$

$$= 6\left(\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right) + c$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left|\sqrt[6]{x}+1\right| + c$$

Theorem (4.6.1)

If an integrand is a rational expression in sin$x$ and cos$x$, then the substitution,

$u = \tan(x/2)$ *for* $-\pi < x < \pi$ will transform the integrand into a rational expression in u with

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\, du$$

Proof

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sec(x/2)} = \frac{1}{\sqrt{1+\tan^2(x/2)}} = \frac{1}{\sqrt{1+u^2}}.$$

$$\sin\left(\frac{x}{2}\right) = \tan\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{u}{\sqrt{1+u^2}}.$$

Then

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2u}{1+u^2}$$

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right) = 1 - \frac{2u}{1+u^2} = \frac{1-u^2}{1+u^2}.$$

Since $x/2 = \tan^{-1} u$, we have $x = 2\tan^{-1} u$ and therefore,

$$dx = \frac{2}{1+u^2}\, du$$

Example (3) Evaluate, $\displaystyle\int \frac{1}{4\sin x - 3\cos x}\, dx$.

Solution

Use the substitution, $u = \tan(x/2)$, then

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\, du$$

$$\int \frac{1}{4\sin x - 3\cos x}\, dx = \int \frac{1}{4\left(\frac{2u}{1+u^2}\right) - 3\left(\frac{1-u^2}{1+u^2}\right)} \left(\frac{2}{1+u^2}\right) du$$

$$= \int \frac{2}{3u^2 + 8u - 3}\, du$$

Using partial fractions, we have

$$\frac{1}{3u^2+8u-3} = \frac{1}{10}\left(\frac{3}{3u-1} - \frac{1}{u+3}\right)$$

$$\int \frac{1}{4\sin x - 3\cos x}\,dx = \frac{1}{5}\int\left(\frac{3}{3u-1} - \frac{1}{u+3}\right)du$$

$$= \frac{1}{5}\left(\ln|3u-1| - \ln|u+3|\right) + c = \frac{1}{5}\ln\left|\frac{3u-1}{u+3}\right| + c$$

$$= \frac{1}{5}\ln\left|\frac{3\tan(x/2)-1}{\tan(x/2)+3}\right| + c\,.$$

Exercises (4.7)

(I) Evaluate the following integrals.

(1) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

(2) $\displaystyle\int x\,\sqrt[3]{x+9}\,dx$

(3) $\displaystyle\int \frac{1}{(x+1)\sqrt{x-2}}\,dx$

(4) $\displaystyle\int \frac{x}{\sqrt[3]{3x+2}}\,dx$

(5) $\displaystyle\int \frac{x+1}{\sqrt[3]{x+4}}\,dx$

(6) $\displaystyle\int \frac{1}{\sqrt[4]{x}+\sqrt[5]{x}}$

(7) $\displaystyle\int e^{3x}\sqrt{1+e^{x}}\,dx$

(8) $\displaystyle\int e^{x}\sqrt{1+e^{3x}}\,dx$

(9) $\displaystyle\int \sin\sqrt{x+4}\,dx$

(10) $\displaystyle\int \cos\sqrt{x+1}\,dx$

(11) $\displaystyle\int_2^3 \frac{x}{(x-1)^4}\,dx$

(12) $\displaystyle\int \frac{x+1}{\sqrt[3]{x+4}}\,dx$

(13) $\displaystyle\int \frac{\sin x}{\cos x\,(\cos x - 1)}\,dx$

(14) $\displaystyle\int \frac{1}{1+\sin x+\cos x}\,dx$

(15) $\displaystyle\int \frac{\sin 2x}{\sin^2 x - 2\sin x - 8}\,dx$

(16) $\displaystyle\int \frac{1}{\sin x - \sqrt{3}\cos x}\,dx$

(II) Prove that:

(1) $\displaystyle\int \sec x\,dx = \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + c\,.$

(2) $\displaystyle\int \csc x\,dx = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + c\,.$

Chapter (5)

Applications Of The Definite Integral

5.1. Area

The area of any region may be considered as the area between a curve of a function and an axis or the area between two curves

5.1.1. Area between curve and axis

Consider that the function $y = f(x)$ is continuous and non negative on $[a, b]$, then the area A of the region bounded by the curve of the function $y = f(x)$ and the *x-axis* over the interval $[a, b]$ is obtained by integrating the element area dA of the vertical rectangle (strip) of width dx and length $f(x)$ as illustrated in Fig. (5.1).

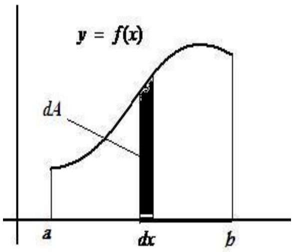


Fig. (5.1)

The element area of this strip is: $dA = f(x)\ dx$.

Hence the area of the region,

$$A = \int_a^b f(x) \quad dx$$

If we consider that the function $x = g(y)$ is continuous and non negative on $[c, d]$, then the area A of the region bounded by the curve of the function $x = g(y)$ and the *y-axis* over the interval $[c, d]$ is illustrated in Fig. (5.2) as

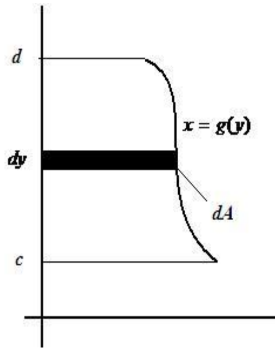


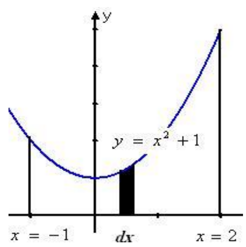
Fig. (5.2)

$$A = \int_c^d g(y) \ dx$$

Example (1) Find the area of the region bounded by the graph of the function $y = x^2 + 1$ and the $x - axis$ from $x = -1$ to $x = 2$.

Solution

As shown in Fig.(5.3), we use a vertical strip of width dx and length $(x^2 - 1)$, then,



$$A = \int_{-1}^{2} (x^2 + 1)\, dx = \left. \frac{x^3}{3} + x \right]_{-1}^{2}$$

$$= \frac{8}{3} + 2 - \frac{-1}{3} - (-1) = 6 \textit{ square units}.$$

Fig.(5.3)

Example (2) Find the area of the region bounded by the graph of the function $y = \cos x$ and the $x-axis$ from $x = 0$ to $x = \pi/2$.

Solution As shown in Fig. (5.4), we use a vertical strip of width dx and length $(\cos x)$, then

$$A = \int_{0}^{\pi/2} \cos x \;\; dx = \sin x \Big]_{0}^{\pi/2} = \sin \pi/2 - \sin 0$$

$$= 1 - 0 = 1 \textit{ square units}.$$

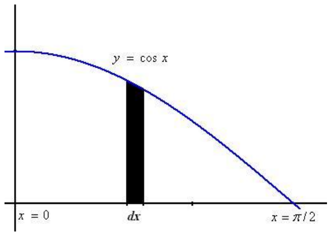


Fig. (5.4)

Example (3) Find the area of the region bounded by the graph of the function $y = x^3$ and the $y-axis$ from $y = 0$ to $y = 8$.

Solution

As shown in Fig. (5.5), we use a horizontal strip of width dy and length $x = \sqrt[3]{y}$, then

$$A = \int_{0}^{8} y^{1/3} \;\; dy = \left. \frac{3y^{4/3}}{4} \right]_{0}^{8} = \frac{3}{4}\sqrt[3]{8^4} = 12 \quad \textit{square units}.$$

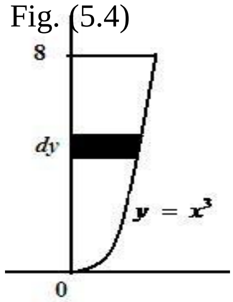


Fig. (5.5)

5.1.2 Area between two curves

If $f_1(x)$ and $f_2(x)$ are continuous functions on the interval $[a, b]$, and if $f_1(x) \geq f_2(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f_1(x)$, and below by $y = f_2(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ can be obtained by considering a vertical rectangle (strip) of width dx.

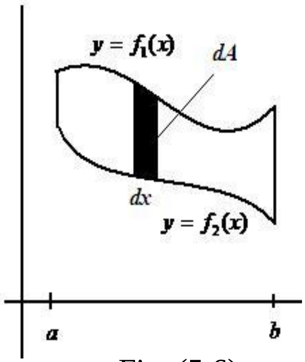


Fig. (5.6)

The length of the strip $f_1(x) - f_2(x)$ as illustrated in Fig. (5.6). The element area of this strip is: $dA = [f_1(x) - f_2(x)]\, dx$. Hence the area of the region is, .

$$A = \int_a^b [f_1(x) - f_2(x)]\; dx$$

Similarly If $g_1(y)$ and $g_2(y)$ are continuous functions on the interval $[c, d]$, and if $g_1(y) \ge g_2(y)$ for all y in $[c, d]$, then the area of the region bounded right by $x = g_1(y)$, and left by $x = g_2(y)$, and from above and below by the line $y = d$, and the line $y = c$ can be obtained by considering a horizontal rectangle (strip) of width dy. The length of the strip $g_1(y) - g_2(y)$ as illustrated in Fig. (5.7).

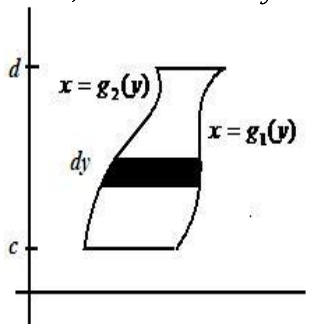


Fig. (5.7)

The element area of this strip is:

$$dA = [g_1(y) - g_2(y)]\, dy .$$

Hence the area of the region is,

$$A = \int_c^d [g_1(y) - g_2(y)]\; dy$$

Example (4) Find the area of the region bounded by the parabola: $y = x^2$ and the line: $y = x + 2$.

Solution

The limits of integration are found by solving the equations of the curve and the straight line to obtain the points of intersection as $x = -1$ *and* $x = 2$.

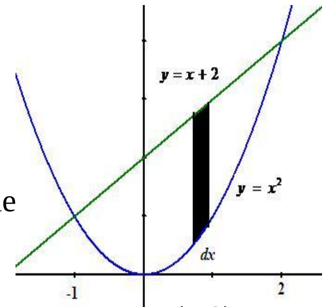


Fig. (5.8)

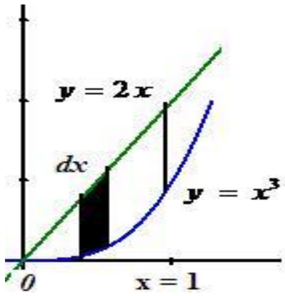
Use vertical strip Fig. (5.8) of width dx and length $[(x + 2) - x^2]$, then

$$A = \int_{-1}^{2} [(x+2) - x^2]\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{10}{3} - \frac{-7}{6} = \frac{9}{2} = 4.5 \quad \textit{square units}.$$

Example (5) Find the area of the region bounded by the graphs: $y = x^3$, $y = 2x$, $x = 0$ *and* $x = 1$.

Solution As shown in Fig. (5.9), we use a vertical strip of width dx and length $(2x - x^3)$, then



$$A = \int_0^1 (2x - x^3)\, dx = \left[x^2 - \frac{x^4}{4}\right]_0^1$$

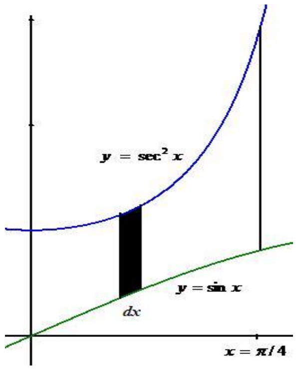
$$= \left[x^2 - \frac{x^4}{4}\right]_0^1 = 1 - \frac{1}{4} = \frac{3}{4} = 0.75 \ \ \textit{square units}.$$

Fig. (5.9)

Example (6) Find the area of the region bounded by the graphs: $y = \sec^2 x$, $y = \sin x$, $x = 0$ *and* $x = \pi/4$.

Solution

As shown in Fig. (5.10), we use a vertical strip of width dx and length $(\sec^2 x - \sin x)$, then



$$A = \int_0^{\pi/4} (\sec^2 x - \sin x)\, dx = \tan x + \cos x \Big]_0^{\pi/4}$$

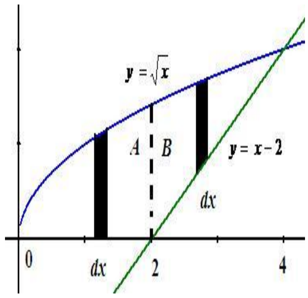
Fig. (5.10)

$$= (1 + \frac{1}{\sqrt{2}}) - (0 + 1) = \frac{1}{\sqrt{2}} \approx 0.707 \ \textit{square units}.$$

Example (7) Find the area of the region in the first quadrant that is bounded by the graphs:

$y = \sqrt{x}$, $y = x - 2$, and the $x-axis$.

Solution



As shown, in Fig. (5.11) we partition the region at $x = 2$ into two sub-regions A and B and evaluate the area of each sub-region separately. We use a vertical strip for each sub-region.

Fig. (5.11)

$\textit{Total area} = \textit{area of the region}\, A + \textit{area of the region}\, B$

$$= \int_0^2 \sqrt{x}\, dx + \int_2^4 [\sqrt{x} - (x - 2)]\, dx = \int_0^2 \sqrt{x}\, dx + \int_2^4 \sqrt{x} - x + 2)\, dx$$

$$= \frac{2}{3} x^{3/2}\Big]_0^2 + \frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x\Big]_2^4$$

$$= \left[\frac{2}{3}(2)^{3/2} - 0\right] - \left[(\frac{2}{3}(4)^{3/2} - 8 + 8) - (\frac{2}{3}(2)^{3/2} - 2 + 4)\right]$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3} \approx 3.333 \ \textit{square units}.$$

Another Solution

We can take the region as one region [Fig. (5.12) if we consider horizontal strip of width dy and length $(y + 2 - y^2)$, then

$$A = \int_0^2 (y + 2 - y^2)\ dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2$$

$$= \left(6 - \frac{8}{3}\right) = \frac{10}{3} \approx 3.333 \ \textit{square units}.$$

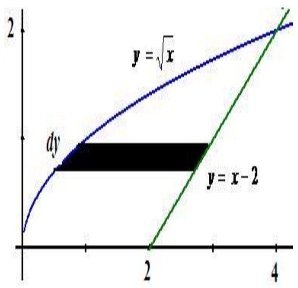


Fig. (5.12)

Example (8) Find the area of the region that is bounded by the graphs: $y = e^x$, $y = \ln x$, $x = 1$ and $x = 2$.

Solution

As shown, in Fig. (5.13) we use a vertical strip of width dx and length $(e^x - \ln x)$, then

$$A = \int_1^2 \left(e^x - \ln x\right)dx = \left(e^x - x\ln x + x\right)_1^2 = 5.284479909$$

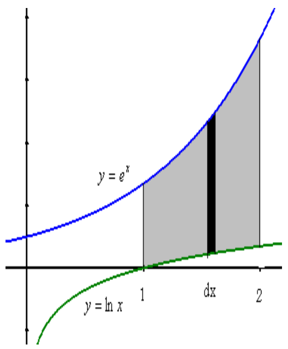


Fig. (5.13)

Example (9) Find the area of the region that is bounded by the graphs: $y = \sinh 3x$, $y = 0$, and $x = 1$.

Solution

As shown, in Fig. (5.14) we use a vertical strip of width dx and length $\sinh 3x$, then

$$A = \int_0^1 \sinh 3x\ dx = \frac{1}{3}\int_0^1 3 \sinh 3x\ dx$$

$$= \frac{1}{3}\left(\cosh 3x\right)_0^1 = \frac{1}{3}\left(10.067662 - 1\right) = 3.022554 \ .$$

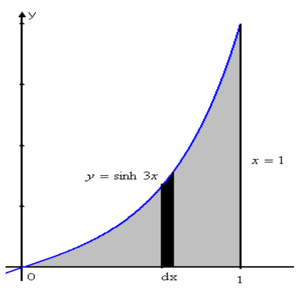


Fig. (5.14)

Exercises (5.1)

(I) Sketch the region bounded by the graphs of the following equations and find its area.

(1) $y = x^2 + 1$, $y = x - 2$, and $x \in [-2, 3]$.

(2) $y = x^2 + 2$, $y = x - 1$, and $x \in [-1, 2]$.

(3) $y = x^2$ and $y = 4x$.

(4) $y = x^2$ *and* $y = x + 6$.

(5) $y = x^2 + 1$ and $y = 5$.

(6) $y = x^2 - 2$ and $y = 2$

(7) $y = 4 - x$, $y = 3x$ and $y = x$.

(8) $y = x^4$ and $y = 8x$.

(9) $y = 1/x^2$, $y = -x^2$ and $x \in [1, 2]$.

(10) $y = 1/x^2$, $y = x^2$ and $y = 2$.

(11) $x = 4y - y^3$ and $x = 0$.

(12) $y = 1/x$, $y = x$ and $x \geq 0$.

(13) $y = x^3 - x$ and $y = 0$.

(14) $y = x^2 - x$, $y = x^2$ and $y = 2$.

(15) $y = \sin x$, $y = \cos x$ and $x \in [0, 2\pi]$.

(16) $y = 8\cos x$, $y = \sec^2 x$, $x = -3\pi/4$ and $x = \pi/3$

(17) $y = \sqrt{x}$, and $y = x^2$.

(18) $y = \sin x$, $y = \cos x$, $x = 0$ and $x = 2\pi$

(19) $y + x^2 = 6$, and $y + 2x - 3 = 0$.

(20) $y = 2\sin x$, $y = \sin 2x$, $x = 0$ and $x = \pi$

(21) $y = x^2 - 1$ and $y = 1 - x^2$.

(22) $y = x^2$ and $y = 2 - x^2$

(23) $y = \ln x$, $y = \ln 2x$, $x = 1$ and $x = 5$

(24) $y = \cosh x$, $x = 0$ and $y = 2$

(25) $y = e^x$, $y = e^{2x}$, and $x = \ln 3$

(26) $y = e^{-x/2}$, $y = e^{x/2}$, and $x = 2\ln 2$.

<u>5.2 Volumes of Solids of Revolution</u>

solids of revolution are solids whose shapes can be generated by revolving plane regions about axis. This axis is called *the axis of rotation.* As illustrated in Fig. (5.15), the solids of revolution are generated from the rotation of the corresponding plane regions about the indicated axis.

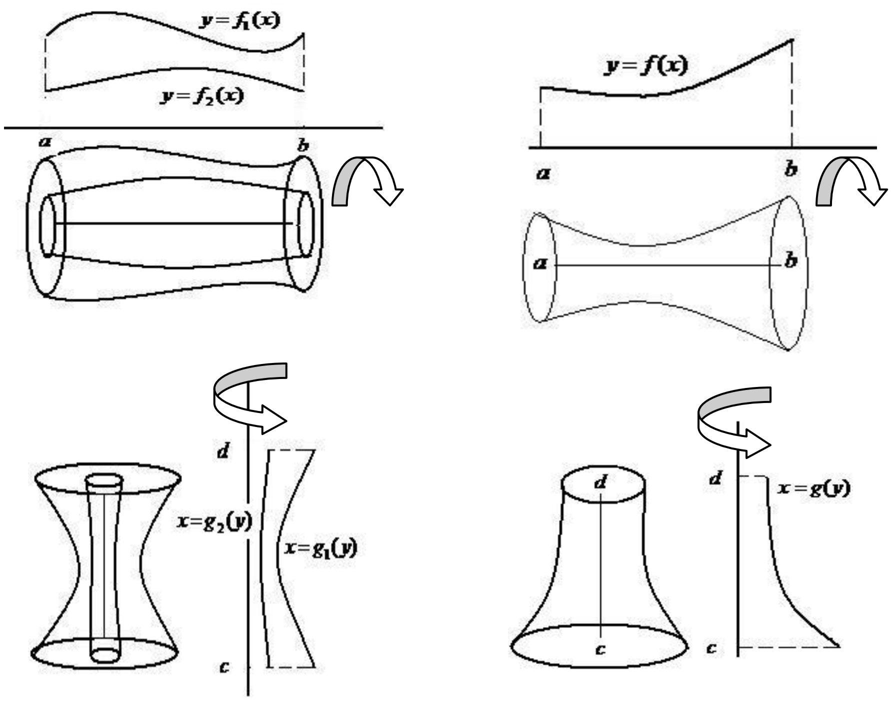


Fig. (5.15)

In this section we shall discuss several methods for finding volumes of the solids of revolution.

Consider a rectangle strip across the plane region, If the volume of the solid generated by revolving this strip about *the axis of rotation* is dv, then the volume of the solid generated by the plane area is ,

$$V = \int_a^b dv$$

According to the solid generated by rectangular strip, we have the following methods.

5.2.1 Volume by Disk

This method is useful when *the axis of rotation* is part of the boundary of the plain area. In this case, we choose the rectangular strip perpendicular to *the axis of rotation.* The solid generated by revolving the rectangular strip about *the axis of rotation* is a **disk** (or cylinder)..

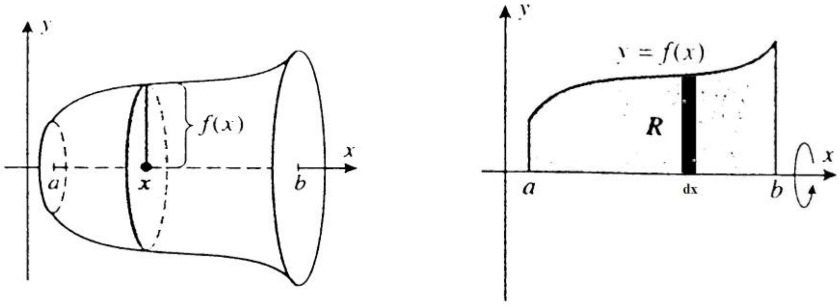
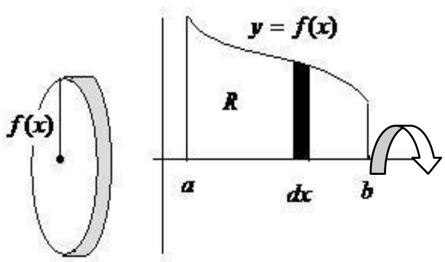


Fig. (5.16)

The element volume of this disk is,

$$dv = \pi\ (radius)^2\ (thickness)\,.$$

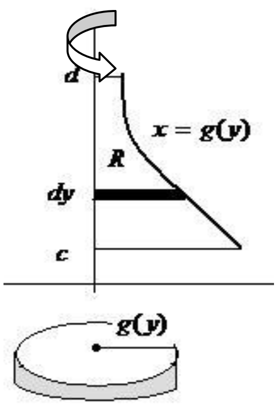


Now let $f(x)$ be continuous for $x \in [a, b]$, and let R be the region bounded above by the graph of $f(x)$ and below the *x-axis* and on the sides by the lines $x = a$ and $x = b$.

Fig. (5.17)

The Volume V of the solid of revolution generated by revolving R about the *x-axis* is,

$$V = \pi \int_a^b f^2(x)\ dx$$



Similarly, if $g(y)$ be continuous for $y \in [c, d]$, and let R be the region bounded from right by the graph of $g(y)$ and from left by the *y-axis* and on the sides by the lines $y = c$ and $y = d$.

The Volume V of the solid of revolution generated by

Fig. (5.18)

revolving R about the *y-axis* is,

$$V = \pi \int_c^d g^2(x)\ dx$$

Example (1) Find the volume of the solid generated by revolving the region bounded by the graphs of $y = x^2 + 1$, the *x-axis* and the lines $x = -1$, and $x = 1$ about the ***x-axis***.

Solution

Use vertical strip of width dx , the radius of the disk $f(x) = x^2 + 1$, then

$$V = \pi \int_{-1}^{1} (x^2 + 1)^2\ dx = \pi \int_{-1}^{1} (x^4 + 2x^2 + 1)\ dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_{-1}^{1} = \pi\, [(\frac{28}{15}) - (-\frac{28}{15})$$

$$= \frac{56}{15}\pi \approx 11.7 \quad \textit{cubic units}.$$

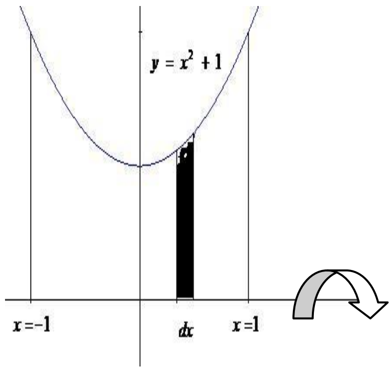


Fig. (5.19)

Example (2) Find the volume of the solid generated by revolving the region bounded by the graphs of the equation $y = \sqrt{x}$, the *x-axis* and the lines $x = 1$ and $x = 4$ about the *x-axis.*

Solution

Use vertical strip of width dx , the radius of the disk $f(x) = \sqrt{x}$, then

$$V = \pi \int_{1}^{4} (\sqrt{x})^2\ dx = \pi \int_{1}^{4} x\ dx = \left. \frac{\pi\, x^2}{2} \right]_{1}^{4}$$

$$= 8\pi - \frac{\pi}{2} = \frac{15\pi}{2} \approx 23.56\ \textit{cubic units}$$

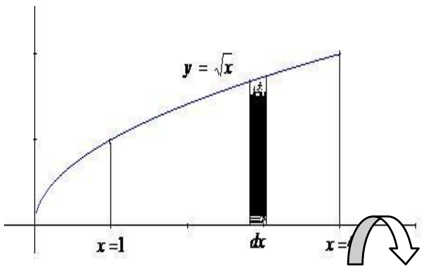


Fig. (5.20)

Example (3) Find the volume of the solid generated by revolving the region bounded by the graphs of $y = e^x$, the *x-axis* and the lines $x = 0$ and $x = 1$ about the *x-axis.*

Solution.

Use vertical strip of width dx, the radius of the disk $f(x) = e^x$, then

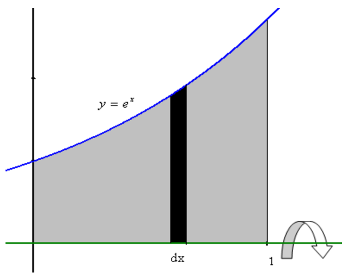


Fig.(5.21)

$$V = \pi \int_0^1 (e^x)^2 \, dx = \pi \int_0^1 e^{2x} \, dx = \frac{\pi}{2} e^{2x} \Big]_0^1 \quad = \frac{\pi}{2}(e^2 - 1) = 10.03590585$$

Example (4) Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 1/x$, $y = 1$, $y = 3$, $x = 0$ about the *y-axis*.

Solution.

Use horizontal strip of width dy , the radius of the disk $x = 1/y$ then

$$V = \pi \int_1^3 \left(\frac{1}{y}\right)^2 dy = \pi \left. \frac{-1}{y} \right)_1^3$$

$$= \pi\,(-1/3 + 1) = 2.094395102$$

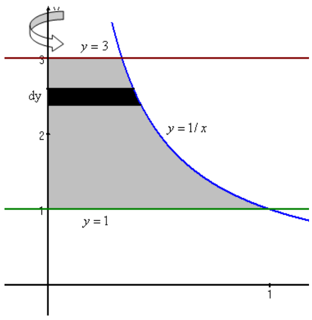


Fig.(5.22)

Example (5) Find the volume of the solid generated by revolving the region bounded by the graphs of $y = e^x$, the *y-axis* and the line $y = 2$ about the *y-axis*.

Solution.

Use horizontal strip of width dy , the radius of the disk $x = \ln y$ then

$$V = \pi \int_1^2 (\ln y)^2 \, dy \quad = \pi \left[x(\ln x)^2 - 2x \ln x + 2x\right]_1^2$$

$$= \pi (2(\ln 2)^2 - 4\ln 2 + 2) = 0.5918377$$

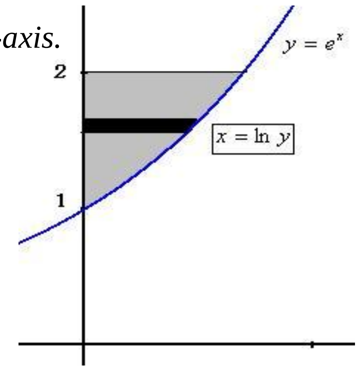


Fig.(5.23)

5.2.2 Volume by Washer

Let us next consider the region R of the type illustrated in Fig. (5.24). If this region is revolved about the *x-axis*, we obtain the solid in the same figure. If $g(x) > 0$ for every $x \in [a, b]$ there is a hole through the solid.

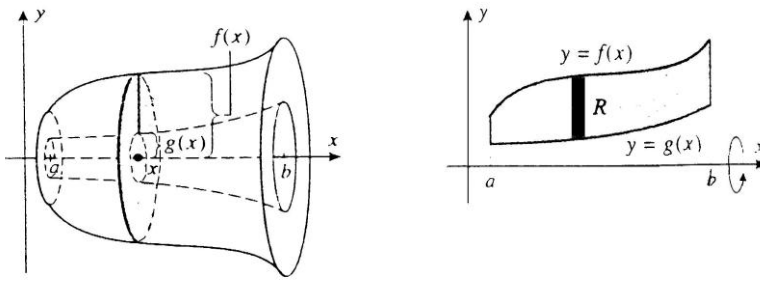


Fig. (5.24)

In this case we can choose the rectangular strip perpendicular to *the axis of rotation.* The solid generated by revolving the rectangular strip about *the axis of rotation* is a **washer** .

The element volume of the washer shown in Fig. (5.24) is,

$$dv = \pi[\,(outer\ \ radius)^2 - (inner\ \ radius)^2\,]\,(thickness).$$

Now let $f_1(x)$ and $f_2(x)$ be continuous for $x \in [a,\, b]$, Fig. (5.25), and suppose that $f_1(x) \geq f_2(x)$ for all $x \in [a,\, b]$.

let R be the region bounded above by the graph of $f_1(x)$, below by the graph of $f_2(x)$, and on the sides by the lines $x = a$ and $x = b$.

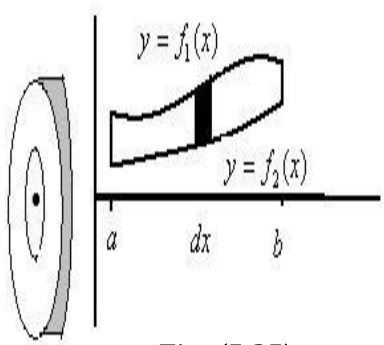


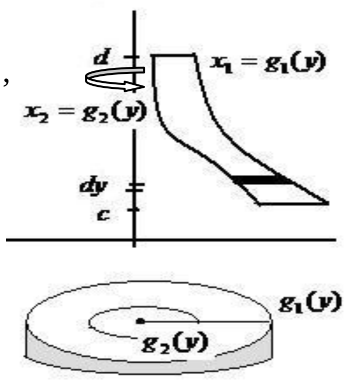
Fig. (5.25)

The Volume V of the solid of revolution generated by revolving R about the *x-axis* is,

$$V = \pi \int_a^b \left[f_1^{\,2}(x) - f_2^{\,2}(x) \right] dx$$

Now let $g_1(y)$ and $g_2(y)$ be continuous for $y \in [c,\, d]$, Fig. (5.26), and suppose that $g_1(y) \geq g_2(y)$ for all $y \in [c,\, d]$.

Let R be the region bounded above by the graph of $g_1(y)$, below by the graph of $g_2(y)$, and on the sides by the lines $y = c$ and $y = d$.



The Volume V of the solid of revolution generated by revolving R about the *y-axis* is, Fig. (5.26)

$$V = \pi \int_c^d \left[g_1^{\,2}(y) - g_2^{\,2}(y) \right] dy$$

Example (6) Find the volume of the solid generated by revolving the region bounded by the graphs of the equation $y = x^2 + 2$, $2y - x - 2 = 0$, and the lines $x = 0$ and $x = 1$ about the *x-axis*.

Solution

Use vertical strip of width dx which generate a washer of *thickness* dx.

The *outer radius* of the washer is $f_1(x) = x^2 + 2$ and the *inner radius*

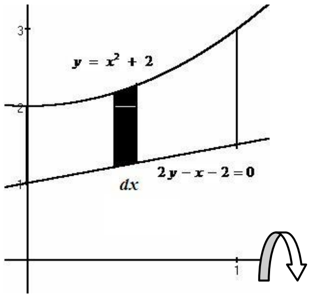


Fig. (5.27)

is $f_2(x) = \frac{1}{2} x + 1$, then,

$$V = \pi \int_0^1 \left[(x^2 + 2)^2 - (\frac{1}{2} x + 1)^2 \right] dx \qquad = \pi \int_0^1 \left[x^4 - \frac{15}{4} x^2 - x + 3 \right] dx$$

$$= \pi \left(\frac{x^5}{5} + \frac{15}{4}\left(\frac{x^3}{3}\right) - \frac{x^2}{2} + 3x \right)_0^1 = \frac{79}{20}\pi \approx 12.4 \; cubic \; units.$$

Example (7) Find the volume of the solid generated by revolving the region bounded by the graphs of the equation $y = x^2 + 1$, *and* $y + x = 3$, about the *x-axis.*

Solution Use vertical strip of width dx which generate a washer of *thickness* dx.

The *outer radius* of the washer is $f_1(x) = -x + 3$ and the *inner radius* is $f_2(x) = x^2 + 1$. The limits of integration may be obtained as the points of intersection of $f(x)$ and $g(x)$, these points are $x = -2$ *and* $x = 1$, then

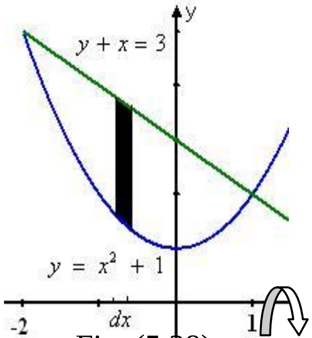


Fig. (5.28)

$$V = \pi \int_{-2}^{1} \left[(-x + 3)^2 - (x^2 + 1)^2 \right] dx \qquad = \pi \int_{-2}^{1} \left[8 - 6x - x^2 - x^4 \right] dx$$

$$= \pi \left(8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right)_{-2}^{1} = \frac{117}{5}\pi \approx 73.5 \;\; cubic \; units.$$

Example (8) Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graphs of the equation $y = \frac{1}{8}x^3$, *and* $y = 2x$, about the *y-axis.*

Solution

Use horizontal strip of width dy which generate a washer of *thickness* dy .

The *outer radius* of the washer is $g_1(y) = 2\sqrt[3]{y}$ and the *inner radius* is $g_2(y) = y/2$.

The limits of integration may be obtained as $x = -2$ *and* $x = 1$, then

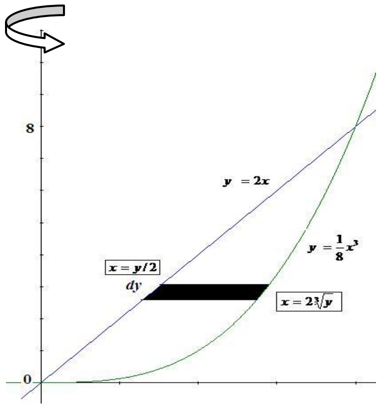


Fig. (5.29)

$$V = \pi \int_0^8 \left[(2\sqrt[3]{y})^2 - (y/2)^2 \right] dy$$

$$= \pi \int_0^8 \left[4y^{2/3} - y^2/4 \right] dy = \pi \left(\frac{12}{5} y^{5/3} - \frac{1}{12} y^3 \right)_0^8 . = \frac{512}{15}\pi \approx 107.2 \ \ cubic\ units$$

Example (9) Find the volume of the solid generated by revolving the region bounded by the graphs: $y = \ln x$, $y = 0$, and $x = c$, about the *y-axis.*

Solution Washer for $0 \le y \le \ln c$

$$V = \pi \int_0^{\ln c} \left[c^2 - (e^y)^2 \right] dy = \pi \left[c^2 y - \frac{1}{2} e^{2y} \right]_0^{\ln c}$$

$$= \pi \left(c^2 \ln c - \frac{1}{2} c^2 + \frac{1}{2} \right)$$

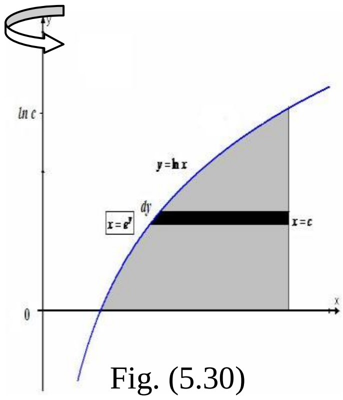


Fig. (5.30)

5.2.3 Volume by Cylindrical Shell

In the preceding sections we found volumes of solids of revolutions by using rectangle strip perpendicular to the *axis of rotation* to produce disk or washer. For certain types of solids, it is difficult to use the above methods, so it is convenient to use rectangle strip parallel to the *axis of rotation* which produce *hollow circular cylinders*, that is, thin **cylindrical shells** of the type illustrated in Fig. (5.29).

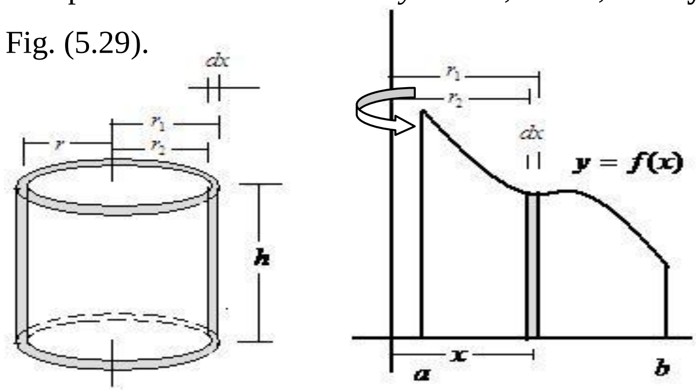


Fig. (5.31)

In Fig. (5.31), r_1 is the *outer radius*, r_2 is the *inner radius*, *h* is the altitude and $dr = r_1 - r_2$ is the *thickness* of the shell. The average radius of the shell is $r = \frac{1}{2}(r_1 + r_2)$. The volume of the shell is,

$$dV = \pi r_1^2 h - \pi r_2^2 h = \pi (r_1^2 - r_2^2) h \quad = \pi (r_1 + r_2)(r_1 - r_2) h$$

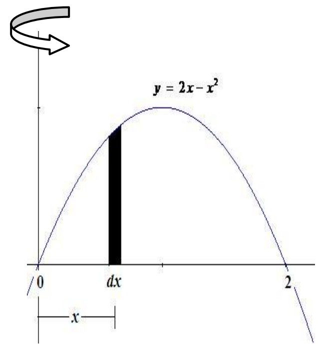
$$= 2\pi \frac{(r_1 + r_2)}{2} h \, (r_1 - r_2) = 2\pi r h dx \, ,$$

which gives us the following general rule

$$dv = 2\pi \ (average\ radius)\ (Altitude)\ (thickness)$$

The ***average radius*** is the distance between the strip and the axis of rotation.
The volume of the solid generated by revolving the region bounded by the graphs of $y = f(x)$, *x-axis*, $x = a$ and $x = b$ about the y-axis is

$$V = 2\pi \int_a^b x \ f(x) \ dx$$



Example (10) Find the volume of the solid generated by revolving the region bounded by the graphs of the equation $y = 2x - x^2$, and the *x-axis* about the *y-axis* Fig. (5.32)

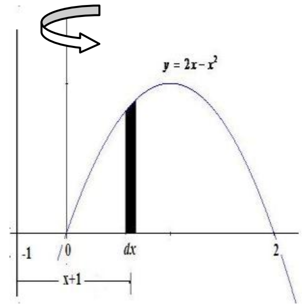
Solution

Use vertical strip of width dx which generate

a cylindrical shell of *thickness* dx and *average radius x*.

$$V = 2\pi\int_0^2 x\,(2x - x^2)\,dx = 2\pi\int_0^2 (2x^2 - x^3)\,dx \quad = 2\pi\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^2 = \frac{8\pi}{3} \approx 8.4$$

Example (11) Find the volume of the solid generated in example (10) if the rotation about the line $x = -3$



Solution

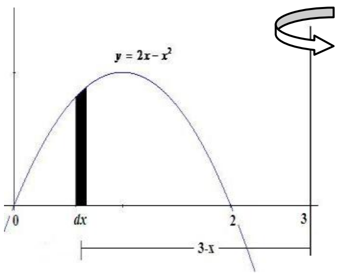
Use vertical strip of width dx which generate

a cylindrical shell of *thickness* dx and *average radius* $x+1$.

$$V = 2\pi\int_0^2 (x+1)\,(2x - x^2)\,dx = 2\pi\int_0^2 (2x + x^2 - x^3)\,dx$$

Fig. (5.33)

$$= 2\pi\left(x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right)_0^2 = \frac{16\pi}{3} \approx 16.8$$



Example (12) Find the volume of the solid generated in example (10) if the rotation about the line $x = 3$

Solution

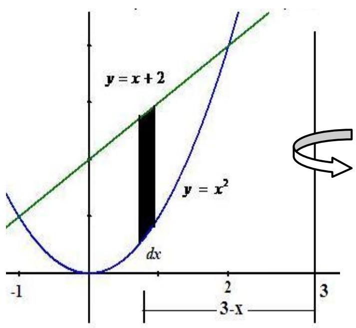
Use vertical strip of width dx which generate

Fig. (5.34)

a cylindrical shell of *thickness* dx and *average radius* $3 - x$.

$$V = 2\pi\int_0^2 (3 - x)\,(2x - x^2)\,dx = 2\pi\int_0^2 (6x - 5x^2 + x^3)\,dx$$

$$= 2\pi\left(3x^2 - \frac{5x^3}{3} + \frac{x^4}{4}\right)_0^2 = \frac{16\pi}{3} \approx 16.8$$



Example (13) Find the volume of the solid generated by revolving the region bounded by the graphs of the equation $y = x^2$, and $y = x+2$ about the line $x = 3$

Fig. (5.35)

Solution

Use vertical strip of width dx which generate a cylindrical shell of *thickness* dx and *average radius* $3 - x$.

$$V = 2\pi\int_{-1}^{2}(3-x)(x+2-x^2)\ dx = 2\pi\int_{-1}^{2}\ (6+x-4x^2+x^3)\ dx$$

$$= 2\pi\left(3x^2 - \frac{5x^3}{3} + \frac{x^4}{4}\right)_0^2 = \frac{16\pi}{3} \approx 16.8$$

Exercises (5.2)

(I) Find the volume of the solid generated by revolving the region bounded by the following curves about the *x-axis.*

(1) $y = x\ ,\quad y = 0 \quad and \quad x = 1.$

(2) $y = 2x\ ,\quad x = 0 \quad and \quad y = 1.$

(3) $y = 1/x\ ,\quad x = 1,\quad x = 3\ ,\quad y = 0\ .$

(4) $y = x^2\ ,\quad y = \sqrt{x}\ .$

(5) $y = x^3,\quad x = -3\ ,\quad y = 0\ .$

(6) $y = x - x^2,\quad y = 0\ .$

(7) $y = x^3,\quad y + x = 10,\quad y = 1$

(8) $y = x^3,\quad and \quad y = 2 - x.$

(9) $y = x^2\ ,\quad y = 2 - x\ .$

(10) $y = x^2,\quad y = 0 \quad and \quad x = 2.$

(11) $y = 2 - 2x^2,\quad y = 1 - x^2\ .$

(12) $y = x^2,\quad x = 0 \quad and \quad y = 1$

(13) $y = x^3\ ,\quad y = -x^2$

(14) $y = \cos x,\quad y = x + 1 \quad and \quad x = \pi/2.$

(15) $y = \sin x\ ,\quad x = \pi/4 \quad x = \pi/2,\quad y = 0\ .$

(16) $y = \sin x,\quad y = 0 \quad and \quad x = \pi/2.$

(17) $y = \cos x$, $y = x + 1$, $x = \pi / 2$

(II) Find the volume of the solid generated by revolving the region bounded by the following curves about the *y-axis.*

(1) $y = 1/x$, $y = 1$, $y = 3$, $x = 0$.

(2) $y = 1/x^2$, $y = 1$, $y = 2$, $x = 0$.

(3) $x + 3y = 6$, $x = 0$, $y = 0$.

(4) $y = 1 - x$, $x = 0$, $y = 0$

(5) $y^2 = x$, $2y = x$.

(6) $y = x^3$, $y = 2$, $x = 0$

(7) $y = x^3$, $y = \sqrt{x}$

(8) $y = x^3$, $y = x$

(9) $y^2 = x$, $y^2 = 2 - x$.

(10) $y = 1 - x^2$, $x = 0$, $y = 0$

(11) $x + y = 1$, $x - y = -1$, $y = 0$, $x = 2$.

(12) $y = 1 + x$, $y = 1 - x$, $x = 0$, $y = 0$

(13) $y = \sqrt{x}$, $y = x^2$

(14) $y = \sqrt{x}$, $y = x$

(15) $y = \sqrt{9 - x^2}$, $x = 0$

(16) $y = \sqrt{9 - x^2}$

(17) $x = y^2 - 4$, $x = 0$, $y = 0$.

(18) $x = 4y - y^2$, $x = 0$

(III) Find the volume of the solid generated by revolving the region bounded by the following curves about the indicated axis of rotation.

(1) $y = x^2$, $y = \sqrt{x}$; about $x = 3$.

(2) $y = x^2$, $y = \sqrt{x}$; about $x = -3$.

(3) $x = y^2$, $x = y + 2$; about $y = 4$.

(4) $x = y^2$, $x = y + 2$; about $y = -2$.

(5) $y = x^3$, $y = 4x$; about $y = 8$.

(6) $y = x^3$, $y = 4x$; about $x = -2$.

(7) $y = x^3$, $y = 4x$; about $x = 4$.

(8) $y = x^3$, $y = 4x$; about $y = -3$.

(9) $x + y = 3, y + x^2 = 3$; about $x = 2$.

(10) $x + y = 3, y + x^2 = 3$; about $y = 1$.

(11) $y = 1 - x^2$, $x - y = 1$; about $y = 3$.

(12) $y = 1 - x^2$, $x - y = 1$; about $y = -5$.

(13) $y = \sqrt{1 - x^2}$, $x + y = 1$; about *x-axis*

(14) $y = \sqrt{1 - x^2}$, $x + y = 1$; about *y-axis*

(15) $x = y^2$, $x = y + 2$; about *y-axis* .

(IV) Find a formula for the volume of the following indicated solid.

(1) A sphere of radius *r*.

(2) A right circular cone of altitude *h* and radius *r*.

(3) A right circular cylinder of altitude *h* and radius *r*.

5.3 Arc Length and Surface Area

5.3.1 Arc Length

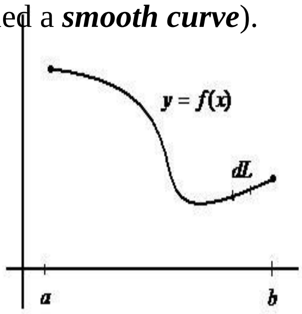
For the surface area of a solid of revolution generated by revolving the curve $y = f(x)$ about an axis of rotation, we must determine the length of the graph of the function $y = f(x)$.

To obtain a suitable formula for the arc length, we consider a smooth curve (A function with a continuous first derivative is said to be ***smooth*** and its graph is called a ***smooth curve***).

Let $y = f(x)$ be a smooth curve, on a closed interval $[a, b]$. Let L is the length of the curve

$y = f(x)$; $x \in [a, b]$.

The length of a small line segment



$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Fig. (5.36)

The length of the curve is defined as,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\ dx$$

Also if $x = g(y)$ be a smooth curve on a closed interval $[c, d]$. The length of a small line segment

$$dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

The length of the curve is defined as,

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\ dy$$

Example (1) Find the arc length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.

Solution

Since $y' = \frac{3}{2}\ x^{1/2}$ which is continuous on $[0,\ 5]$. So $y = x^{3/2}$ is smooth curve on $[0,\ 5]$, and,

$$L = \int_0^5 \sqrt{1 + \frac{9}{4}x}\ dx = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^5 = \frac{335}{27} \approx 12.4$$

Example (2) Find the arc length of the curve $x = y^{3/2} - 1$ from $y = 0$ to $y = 4$.

Solution

Since $\frac{dx}{dy} = \frac{3}{2}\ y^{1/2}$ which is continuous on $[0,\ 4]$. So $x = y^{3/2} - 1$ is smooth curve on $[0,\ 4]$, and,

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}y}\ dy = \left[\frac{8}{27}\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^4 = 9.0734$$

Example (3) Find the arc length of the curve $y = \cosh x$ from $x = 0$ to $x = 1$.

Solution

Since $y' = \sinh x$ which is continuous on $[0, 1]$. So $y = \cosh x$ is smooth curve on $[0, 1]$, and,

$$L = \int_0^1 \sqrt{1 + \sinh^2 x}\; dx = \int_0^1 \cosh x \, dx = \left[\sinh x\right]_0^1 = \sinh(1)$$

$$= \frac{e^1 - e^{-1}}{2} = 1.1752$$

Example (4) Find the arc length of the curve $24xy = x^4 + 48$ from $x = 2$ to $x = 4$.

Solution

Since $y' = \dfrac{x^4 - 16}{8x^2}$ which is continuous on $[2, 4]$. So $24xy = x^4 + 48$ is smooth curve on $[2, 4]$, and,

$$L = \int_2^4 \sqrt{\frac{1}{64}\left(\frac{x^4 + 16}{x^2}\right)^2}\; dx = \frac{1}{8}\int_2^4 \left(x^2 + \frac{16}{x^2}\right) dx$$

$$= \frac{1}{8}\left(\frac{x^3}{3} - \frac{16}{x}\right)_2^4 = \frac{17}{6} \approx 2.83 .$$

5.3.2. Surface area of Solid of Revolution

A surface of revolution is a surface that is generated by revolving a curve about an axis that lies in the same plane as the curve.

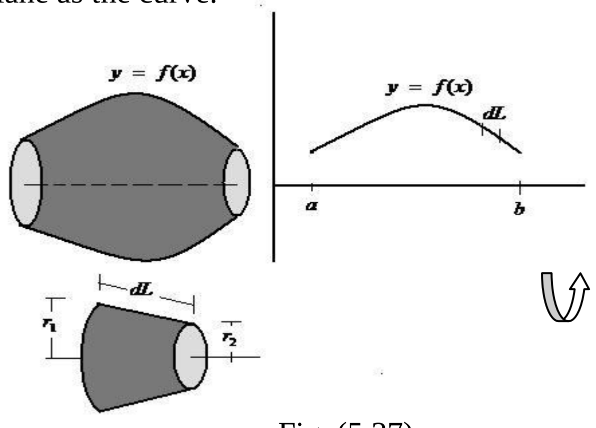


Fig. (5.37)

Let f be a function that is non-negative throughout a closed interval $[a, b]$. If the graph of f is revolved about the *x-axis*, a surface of revolution is generated.

Suppose that a smooth, nonnegative function $y = f(x)$ on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the *x-axis* .

Consider an element length dL of $f(x)$ is revolved about the *x-axis,* then the surface generated is a frustum of a cone having base radii r_1 and r_2 and slant height dL. It can be shown that the surface area is

$$dS = \pi(r_1 + r_2)\, dL = 2\pi \left(\frac{r_1 + r_2}{2}\right) dL$$

that is,

$$dS = 2\pi\ (\textit{average radius})\ (\textit{slant height}).$$

Then,

$$\boxed{S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\ dx.}$$

Now, if $x = g(y)$ is smooth and nonnegative function on $[c, d]$, then the area of the surface generated by revolving the graph of $x = g(y)$ between $y = c$ and $y = d$ about the *y-axis* is,

$$\boxed{S = 2\pi \int_c^d g(y) \sqrt{1 + [g'(y)]^2}\ dy.}$$

Example (5) Find the area of the surface that is generated by revolving the arc of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the *x-axis.*

Solution $y = x^3$, $y' = 3x^2$, then

$$S = 2\pi \int_a^b y \sqrt{1 + [y']^2}\ dx = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\ dx$$

$$= \frac{2\pi}{36} \cdot \frac{2}{3} (1 + 9x^4)^{3/2} \Big]_0^1 \approx 3.563$$

Example (6) Find the area of the surface that is generated by revolving the curve $y = 2\sqrt{x}$ between $x = 1$ and $x = 2$ about the *x-axis.*

Solution

$$y = 2\sqrt{x}\ , \qquad y' = \frac{1}{\sqrt{x}}.$$

$$S = 2\pi \int_a^b y\sqrt{1+[y']^2}\, dx = 2\pi \int_1^2 2\sqrt{x}\sqrt{1+\frac{1}{x}}\, dx$$

$$= 4\pi \int_1^2 \sqrt{x+1}\, dx = 4\pi\, \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\,(3\sqrt{3}-2\sqrt{2}) \approx 19.8$$

Example (7) Find the area of the surface that is generated by revolving the arc of the curve $x = y^3$ between $y = 0$ and $y = 1$ about the *y-axis.*

Solution

$$x = y^3, \qquad \frac{dx}{dy} = 3y^2, \text{ then}$$

$$S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_0^1 y^3\sqrt{1+9y^4}\, dy$$

$$= \frac{2\pi}{36}\cdot\frac{2}{3}\,(1+9y^4)^{3/2}\Big]_0^1 \approx 3.563$$

Example (8) Find the area of the surface that is generated by revolving the arc of the curve $y^2 + 4x = 2\ln y$ between $y = 1$ and $y = 3$ about the *y-axis.*

Solution

$$x = \frac{1}{4}\left(2\ln y - y^2\right), \qquad \frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{y} - y\right), \text{ then}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}\left(\frac{1}{y^2} - 2 + y^2\right) = \frac{1}{4}\left(\frac{1}{y^2} + 2 + y^2\right)$$

$$= \frac{1}{4}\left(\frac{1}{y} + y\right)^2 = \left(\frac{1+y^2}{2y}\right)^2$$

$$S = 2\pi \int_1^3 y\left(\frac{1+y^2}{2y}\right) dy = \pi \int_1^3 (1+y^2)\, dy$$

$$= \pi\,(y + y^3/3)\Big]_1^3 = \pi(12 - 4/3) \approx 33.5$$

Exercises (5.3)

(I) In the following problems, find the arc length of the following curves from the point A to the point B.

(1) $y = x^{3/2}$; A (0, 0) , B (5, $5\sqrt{5}$)

(2) $y = 3x^{3/2} - 1$ A (0, -1) , B (1, 2)

(3) $x = 3y^{3/2} - 1$; A (-1, 0) , B (23, 4)

(4) $y = x^3/12 + 1/x$ A (1, 13/12) , B (2, 7/6)

(5) $24xy = x^4 + 48$; A (2, 4/3) , B (4, 7/6)

(6) $y = (x^6 + 8)/(16x^2)$ A (2, 9/8) , B (3, 737/144)

(7) $27y^2 = 4(x-2)^3$; A (2, 0) , B (11, $6\sqrt{3}$)

(8) $y = 5 - \sqrt{x^3}$; A (1, 4) , B (4, -3)

(9) $(y+1)^2 = (x-4)^3$; A (5, 0) , B (8,7)

(10) $y = 6\sqrt[3]{x^2} + 1$; A(-1, 7) , B (-8, 25)

(11) $y = 6x^{2/3} + 1$; A (1, 7) , B (8, 25)

(12) $y = (1/3)\sqrt{(x^2+2)^3}$; A(0, $2\sqrt{2}$/3) , B (3, $\sqrt{11^3}$/3)

(13) $y = x^3/12 + 1/x$; A(1, 13/12) , B (2, 7/6)

(14) $x = y^3/3 + 1/4y$; A(7/12, 1) , B (109/12, 3)

(15) $y + 1/4x + x^3/12 = 0$; A (1, 13/12) , B (2, 7/6)

(16) $x = (1/3)\sqrt{y^3} - \sqrt{y}$; A(-2/3, 1) , B (6, 9)

(17) $30xy^3 - y^8 = 15$; A (8/15, 1) , B (271/240, 2)

(18) $x = y^4/4 + 1/(8y^2)$; A(3/8, 1) , B (129/32, 2)

(19) $x = y^4/16 + 1/2y^2$; A (9/16, 1) , B (9/8, 2)

(II) Find the area of the surface generated by revolving the following curves about the *x-axis* .

(1) $4x = y^2$; A (0, 0) , B (1, 2)

(2) $y = 7x$; A (0, 0) , B (1, 7)

(3) $y = x^3$; A (1, 1) , B (2, 8)

(4) $x = \sqrt{y}$, A (1, 1) , B (2, 4)

(5) $8y = 2x^4 + x^{-2}$; A (1, 3/8) , B (2, 129/32)

(6) $y = \sqrt{x}$, A (1, 1) , B (4, 2)

(7) $y = 2\sqrt{x+1}$; A (0, 2) , B (3, 4)

(8) $y = \sqrt{4 - x^2}$, A (-1, $\sqrt{3}$) , B (1, $\sqrt{3}$)

(III) Find the area of the surface generated by revolving the following curves about the y-*axis*

.

(1) $y = 2\sqrt[3]{x}$; A (1, 2) , B (8, 4)

(2) $x = y^3/3$; A (0, 0) , B (1/3, 1)

(3) $x = 4\sqrt{y}$; A (4, 1) , B (12, 9)

(4) $x = y^{3/2}/3 - y^{1/2}$; A (0, 0) , B (-2/3, 1)

(5) $y = \sqrt{25 - x^2}$; A (-3, 4) , B (3, 4)

(6) $x = 2\sqrt{4 - y}$; A (4, 0) , B (2, 15/4)

(7) $y = \frac{1}{3}(x^2 + 2)^{3/2}$; A $(0, 2\sqrt{2}/3)$, B $(3, 11\sqrt{11}/3)$

(8) $x = \sqrt{2y - 1}$; A (1/2, 5/8) , B (1, 1)

Chapter (6)

L'Hopital's Rule and Improper Integrals

This chapter is divided into two parts, the first part is concerned with different types of indeterminate forms and their evaluation through a method known as *L'Hopital's rule* and the second part is concerned with *improper integrals*, their types, the convergence and divergence features of them and how to evaluate such types of integrals in some cases.

6.1 Indeterminate Forms and L'Hopital's Rule

We shall consider here several indeterminate forms together with their evaluation.

6.1.1 The Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Let $f(x)$ and $g(x)$ be two functions which are continuous at $x = a$ and either

$$f(a) = g(a) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$$

Then the expression $\lim_{x\to a} \frac{f(x)}{g(x)}$ for both cases cannot be evaluated directly since this leads to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which are known as *indeterminate forms.*

Theorem (6.1.1) "L'Hopital's Rule"

Let $f(a) = g(a) = 0$. If $f'(a)$ and $g'(a)$ exist such that $g'(a) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

If $g'(a) \neq 0$, then from the limit theorems it follows that:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Remark : For the case $\frac{\infty}{\infty}$, L'Hopital's rule states that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the last limit exists or is infinite. It must be noted that L'Hopital's rule can be applied several times as far as we still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and the rule ceases to be applied when either the numerator or the denominator has a finite nonzero limit. Also, it must be noted that the rule is applied if *a* is replaced by $a^+,\ a^-,\ \pm\infty$.

Example (1) Evaluate the following limits,

$(i)\ \lim_{x\to 0}\frac{\sin x}{x}$ $\quad (ii)\ \lim_{x\to 0}\frac{x^2-x}{x^4-5x^2+2x}$ $\quad (iii)\ \lim_{x\to 0}\frac{1-\cos x}{x^4+x^2}$

$(iv)\lim_{x\to 0}\frac{1-2e^x+e^{2x}}{x+\sin x}$ $\quad (v)\ \lim_{x\to 0}\frac{1-2e^x+e^{2x}}{x-\sin x}$ $\quad (vi)\ \lim_{x\to 0}\frac{\sin^{-1}x}{x}$.

Solution

$(i)\ \lim_{x\to 0}\frac{\sin x}{x}=\left(\frac{0}{0}\right)$. Then, $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1.$

$(ii)\ \lim_{x\to 0}\frac{x^2-x}{x^4-5x^2+2x}=\left(\frac{0}{0}\right)$. Then,

$$\lim_{x\to 0}\frac{x^2-x}{x^4-5x^2+2x}=\lim_{x\to 0}\frac{2x-1}{4x^3-10x+2}=-\frac{1}{2}\ .$$

$(iii)\ \lim_{x\to 0}\frac{1-\cos x}{x^4+x^2}=\left(\frac{0}{0}\right)$. Then, $\lim_{x\to 0}\frac{1-\cos x}{x^4+x^2}=\lim_{x\to 0}\frac{\sin x}{4x^3+2x}=\left(\frac{0}{0}\right)$

Hence we use the rule another time, $\lim_{x\to 0}\frac{1-\cos x}{x^4+x^2}=\lim_{x\to 0}\frac{\cos x}{12x^2+2}=\frac{1}{2}$

$(iv)\ \lim_{x\to 0}\frac{1-2e^x+e^{2x}}{x+\sin x}=\left(\frac{0}{0}\right)$. Then,

$$\lim_{x\to 0}\frac{1-2e^x+e^{2x}}{x+\sin x}=\lim_{x\to 0}\frac{-2e^x+2e^{2x}}{1+\cos x}=\frac{0}{2}=0\ .$$

$(v)\lim_{x\to 0}\frac{1-2e^x+e^{2x}}{x-\sin x}=\left(\frac{0}{0}\right)$. Then,

Use the rule another time, $\lim_{x\to 0}\frac{1-2e^x+e^{2x}}{x-\sin x}=\lim_{x\to 0}\frac{-2e^x+2e^{2x}}{1-\cos x}=\left(\frac{0}{0}\right)$.

$$\lim_{x\to 0} \frac{1-2e^x+e^{2x}}{x-\sin x} = \lim_{x\to 0} \frac{-2e^x+4e^{2x}}{\sin x} = \frac{2}{0} = \infty.$$

(vi) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = \left(\frac{0}{0}\right)$. Then, $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{1/\sqrt{1-x^2}}{1} = 1.$

Example (2) Evaluate the following limits,

(i) $\lim_{x\to\infty} \frac{x^2}{e^x}$ (ii) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ (iii) $\lim_{x\to\frac{\pi}{2}^-} \frac{2\tan x}{1+\sec x}$

(iv) $\lim_{x\to\infty} \frac{e^{2x}}{\ln x}$ (v) $\lim_{x\to 0} \frac{2^x-3^x}{x}$ (vi) $\lim_{x\to\infty} \frac{3-3^x}{5-5^x}$

Solution

(i) $\lim_{x\to\infty} \frac{x^2}{e^x} = \left(\frac{\infty}{\infty}\right)$. Then, $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \left(\frac{\infty}{\infty}\right)$.

Hence we use the rule another time, $\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0.$

(ii) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \left(\frac{\infty}{\infty}\right)$. Then, $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$

(iii) $\lim_{x\to\frac{\pi}{2}^-} \frac{2\tan x}{1+\sec x} = \left(\frac{\infty}{\infty}\right)$. Then, $\lim_{x\to\frac{\pi}{2}^-} \frac{2\tan x}{1+\sec x} = \lim_{x\to\frac{\pi}{2}^-} \frac{2\sec^2 x}{\sec x\tan x}$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{2\sec x}{\tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{2/\cos x}{\sin x/\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{2}{\sin x} = 2.$$

(iv) $\lim_{x\to\infty} \frac{e^{2x}}{\ln x} = \left(\frac{\infty}{\infty}\right)$. Then, $\lim_{x\to\infty} \frac{e^{2x}}{\ln x} = \lim_{x\to\infty} \frac{2e^{2x}}{1/x} = \lim_{x\to\infty} 2xe^{2x} = \infty.$

(v) $\lim_{x\to 0} \frac{2^x-3^x}{x} = \left(\frac{0}{0}\right)$. Then,

$$\lim_{x\to 0} \frac{2^x-3^x}{x} = \lim_{x\to 0} \frac{2^x\ln 2-3^x\ln 3}{1} = \ln 2-\ln 3.$$

(vi) $\lim_{x\to\infty} \frac{3-3^x}{5-5^x} = \lim_{x\to\infty} \frac{-3^x\ln 3}{-5^x\ln 5} = \left(\frac{\ln 3}{\ln 2}\right)\lim_{x\to\infty}\left(\frac{3}{5}\right)^x = \left(\frac{\ln 3}{\ln 2}\right)(0) = 0$

6.1.2 Other Indeterminate Forms

There are a number of indeterminate forms other than $\frac{0}{0}$ or $\frac{\infty}{\infty}$ such as $\infty-\infty$, $0\times\infty$, 0^0, ∞^0 and 1^∞. These can be evaluated by transforming them to the form of a quotient and then applying L'Hopital's rule. Each case will investigated separately as follows

1. Case of $\infty-\infty$

This case results from $\lim_{x\to a}[f(x)-g(x)]$ where $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$. We treat this by trying to transform it to one of the previous forms.

Example (3) Evaluate $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

Solution

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left(\frac{\sin x-x}{x\sin x}\right)=\lim_{x\to 0}\left(\frac{\cos x-1}{x\cos x+\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{-\sin x}{-x\sin x+2\cos x}\right)=\frac{0}{2}=0\ .$$

2. Case of $0\times\infty$

This case results from $\lim_{x\to a} f(x)\,g(x)$ where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\pm\infty$. We treat this case by writing $f(x)\,g(x)$ as $\frac{f(x)}{1/g(x)}$ giving rise to $\frac{0}{0}$ or as $\frac{g(x)}{1/f(x)}$ giving rise to $\frac{\infty}{\infty}$.

Example (4) Evaluate

(i) $\lim_{x\to 0^+} x\ln x$ (ii) $\lim_{x\to 0^+} x\cot x$

Solution

(i) $\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0.$

(ii) $\lim_{x\to 0^+} x\cot x=\lim_{x\to 0^+}\frac{x}{\tan x}=\lim_{x\to 0^+}\frac{1}{\sec^2 x}=\frac{1}{1}=1.$

3. For other cases 0^0, ∞^0 and 1^∞

These cases results from $\lim_{x\to a} f(x)^{g(x)}$ where $\lim_{x\to a} f(x) = 0 \; or \; \infty \; or \; 1$ and $\lim_{x\to a} g(x) = 0 \; or \; 0 \; or \; \infty$ respectively.

We treat this by writing $y = f(x)^{g(x)}$ and then taking logarithms we get: $\ln y = g(x) \ln f(x)$, then $\lim_{x\to a} \ln y = \lim_{x\to a}\left(g(x) \ln f(x)\right) = A$

where A is constant and hence, $\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^A$

Example (5) Evaluate $\lim_{x\to 0^+} x^x$

Solution

Let $y = x^x$, then $\ln y = x \ln x$ and

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} y = e^0 = 1$$

Example (6) Evaluate $\lim_{x\to 0^+} x^{1/x}$

Solution

Let $y = x^x$, then $\ln y = x \ln x$ and

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

Example (7) Evaluate $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x$

Solution

Let $y = \left(1 + \frac{3}{x}\right)^x$, then $\ln y = x \ln\left(1 + \frac{3}{x}\right)$ and $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 + \frac{3}{x}\right)$

$$= \lim_{x\to\infty} \frac{\ln\left(1 + \frac{3}{x}\right)}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{\left(1 + \frac{3}{x}\right)} \cdot \frac{-3}{x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{3}{\left(1 + \frac{3}{x}\right)} = 3 .$$ Then $\lim_{x\to\infty} y = e^3$

Exercises (6.1)

Evaluate the following limits,

(1) $\lim_{x\to 0}\frac{\tan 2x}{5x}$ (2) $\lim_{x\to 3}\frac{x^3-27}{x-3}$ (3) $\lim_{x\to 2}\frac{x^5-32}{x-2}$

(4) $\lim_{x\to 0}\frac{3x-\sin x}{x}$ (5) $\lim_{x\to 1}\frac{x^{\sqrt{2}}-1}{x-1}$ (6) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

(7) $\lim_{x\to 0}\frac{x^2-4x+3}{x^2-2x-3}$ (8) $\lim_{x\to 0}\frac{\cos x-1}{e^x-x-1}$ (9) $\lim_{x\to\pi/2}\frac{\cos x}{2x-\pi}$

(10) $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$ (11) $\lim_{x\to 0}\frac{x-\tan^{-1}x}{x\sin x}$ (12) $\lim_{x\to 0^+}\left(\frac{3x+1}{x}-\frac{1}{\sin x}\right)$

(13) $\lim_{x\to 0}\frac{\sin^{-1}3x}{\sin^{-1}x}$ (14) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$ (15) $\lim_{x\to 0}\frac{x\cos x-e^{-x}}{x^2}$

(16) $\lim_{x\to 0}\frac{\sin x}{x^2}$ (17). $\lim_{x\to 0}\frac{2e^x-3x-e^{-x}}{x^2}$ (18) $\lim_{x\to 0}\frac{\sin x^2}{x}$

(19). $\lim_{x\to 0^+}\frac{\ln\sin x}{\ln\sin 3x}$ (20) $\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x}$ (21) $\lim_{x\to\infty}\frac{x^{3/2}+5x-4}{x\ln x}$

(22) $\lim_{x\to 0}\frac{x\sin x}{1-\cos x}$ (23) $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ (24) $\lim_{x\to\pi/2}\frac{\ln(\csc x)}{(x-\pi/2)^2}$

(25) $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^{2x}$ (26) $\lim_{x\to 0^+}(\ln 2x-\ln(1+x))$ (27) $\lim_{x\to 0^+}x^{\sin x}$

(28) $\lim_{x\to 0^+}(\ln x-\ln\sin x)$ (29) $\lim_{x\to 0^+}\frac{\ln(x-1)}{(x-2)^2}$ (30) $\lim_{x\to 0}\frac{x\,2^x}{2^x-1}$

(31) $\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$ (32) $\lim_{x\to\infty}(1+2x)^{1/2\ln x}$ (33) $\lim_{x\to 0}\frac{e^x-2^x}{x}$

(34) $\lim_{x\to\infty}(\sinh x-x)$ (35) $\lim_{x\to 0}(1-3x)^{1/x}$ (36) $\lim_{x\to\infty}x^2\,2^{-x}$

(37) $\lim_{x\to\infty}\frac{\sqrt{10x+1}}{\sqrt{x+1}}$ (38) $\lim_{x\to 1}\frac{x-1}{\ln x-\sin\pi x}$ (39) $\lim_{x\to(\pi/2)^-}\tan x\ln\sin x$

(40) $\lim_{x\to 0^+}(x+\cos 2x)^{\csc 3x}$ (41) $\lim_{x\to 0^+}(1+3x)^{\csc x}$ (42) $\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x$

(43) $\lim_{x\to 0^+}(1+2x)^{\cot x}$ (44) $\lim_{x\to 0}(e^x+x)^{1/x}$ (45) $\lim_{x\to\infty}(\sinh x-x)$

(46) $\lim_{x\to\infty}(x)^{1/\ln x}$ (47) $\lim_{x\to 0^+}(\cot^2 x-\csc^2 x)$ (48) $\lim_{x\to 1^+}(x)^{1/(x-1)}$

6.2 Improper Integrals

The definite integral $\int_a^b f(x)\,dx$ has a finite value when a and b are finite on $[a, b]$. Such integral are said to be *proper*, but if either a or b is infinite or $f(x)$ is infinite on $[a, b]$, the resulting integrals are said to be *improper*. The treatment of improper integrals is carried through limiting processes. If the limit exists, the improper integral is said to

be convergent and the limit is the value of the improper integral. If the limit does not exist, the improper integral is said to be divergent. Two types of improper integrals can appear, the first type are integrals with infinite limits of integration and the second type are integrals with infinite integrands.

6.2.1 Improper Integrals of The First Type

The improper integrals of the first type appears in either one of the following forms:

1. $\int_{-\infty}^{b} f(x)\,dx$, where $f(x)$ is continuous on $(-\infty, b]$.

The treatment of this integral is carried through the following limiting process:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{r \to -\infty} \int_{r}^{b} f(x)\,dx$$

2. $\int_{a}^{\infty} f(x)\,dx$, where $f(x)$ is continuous on $[a, \infty)$.

The treatment of this integral is carried through the following limiting process:

$$\int_{a}^{\infty} f(x)\,dx = \lim_{r \to \infty} \int_{a}^{r} f(x)\,dx$$

3. $\int_{-\infty}^{\infty} f(x)\,dx$, where $f(x)$ is continuous on $(-\infty, \infty)$.

The treatment of this integral is carried through the following limiting process:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{r_1 \to -\infty \\ r_2 \to \infty}} \int_{r_1}^{r_2} f(x)\,dx$$

If the integral does not exists, but $\lim_{r\to\infty}\int_{-r}^{r} f(x)\,dx$ exists, we say that the *principal value* of the integral exists.

Example (1)

Determine whether the following improper integrals converge or diverge

$(i)\ \int_{-\infty}^{-1}\frac{dx}{x^2}$ $\qquad (ii)\ \int_{0}^{1}\frac{dx}{1+x^2}$ $\qquad (iii)\ \int_{-\infty}^{\infty}\sinh x\,dx$

Solution

$$(i)\ \int_{-\infty}^{-1}\frac{dx}{x^2} = \lim_{r\to-\infty}\int_{r}^{-1}\frac{dx}{x^2} = \lim_{r\to-\infty}\left(\frac{-1}{x}\right)_r^{-1} = \lim_{r\to-\infty}\left(1+\frac{1}{r}\right) = 1.$$

Therefore the integral converges and is equal to 1 (see Fig. (6.1)).

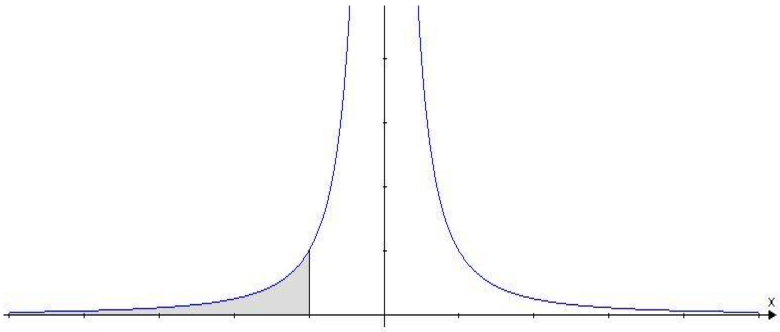


Fig. (6.1)

$(ii)\ \int_{0}^{\infty}\frac{dx}{1+x^2} = \lim_{r\to\infty}\int_{0}^{r}\frac{dx}{1+x^2} = \lim_{r\to\infty}\left(\tan^{-1}x\right)_0^r = \lim_{r\to\infty}\tan^{-1}r = \frac{\pi}{2}$ Therefore the integral converges and is equal to $\pi/2$ (see Fig. (6.2)).

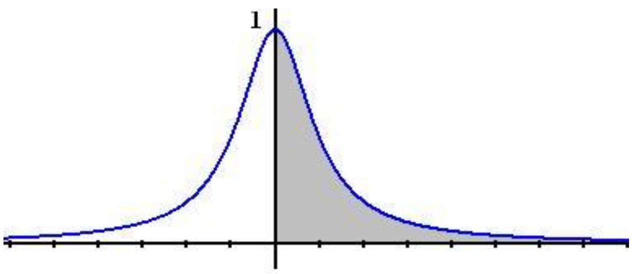


Fig. (6.2).

$$(iii)\int_{-\infty}^{\infty}\sinh x\,dx = \lim_{\substack{r\to\infty\\ t\to-\infty}}\int_{t}^{r}\sinh x\,dx = \lim_{\substack{r\to\infty\\ t\to-\infty}}\left(\cosh x\right)_t^r = \lim_{\substack{r\to\infty\\ t\to-\infty}}\left(\cosh t - \cosh r\right)$$

The limit does not exits and therefore the integral diverges (see Fig. (6.3)). Therefore, the principal value of the integral exists. This means that when approaching $-\infty$ and ∞ in different arbitrary manners, the integral diverges but when the approaches are in qual manners. the principal value of the integral converges.

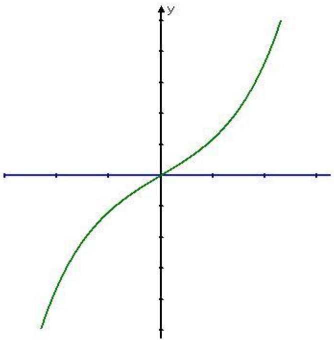


Fig. (6.3).

<u>Theorem (6.2.1) (Domination Comparison Test)</u>

Let $f(x)$ and $g(x)$ be continuous and let $0 \le f(x) \le g(x)$ for all $x \in [a, \infty)$. Then

If $\int_a^\infty g(x)\,dx$ converges, $\int_a^\infty f(x)\,dx$ also converges.

If $\int_a^\infty f(x)\,dx$ diverges, $\int_a^\infty g(x)\,dx$ also diverges.

Example (2)

Show that $\int_1^\infty e^{-x^2}\,dx$ converges

Solution

It is clear that $0 \le e^{-x^2} \le e^{-x} \quad \forall\ x \in [1, \infty)$ and,

$$\int_1^\infty e^{-x}\,dx = \lim_{r\to\infty}\int_1^r e^{-x}\,dx = \lim_{r\to\infty}\left(-e^{-x}\right)_1^r = \lim_{r\to\infty}\left(-e^{-r}+e^{-1}\right) = \frac{1}{e}\ .$$

This means that $\int_1^\infty e^{-x}\,dx$ converges, and therefore $\int_1^\infty e^{-x^2}\,dx$ converges

Example (3)

Show that $\int_1^\infty \frac{1+x}{x^2}\,dx$ diverges

Solution

$$\int_1^{\infty} \frac{1+x}{x^2}\ dx = \int_1^{\infty} \frac{1}{x^2}\ dx + \int_1^{\infty} \frac{1}{x}\ dx$$

Since $\dfrac{1+x}{x^2} > \dfrac{1}{x} \quad \forall\ x \in [1, \infty)$ and,

$$\int_1^{\infty} \frac{1}{x}\ dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x}\ dx = \lim_{t \to \infty} (\ln x)_1^r = \lim_{t \to \infty} (\ln r - \ln 1) = \infty \text{ (diverges).}$$

Then $\displaystyle\int_1^{\infty} \frac{1+x}{x^2}\ dx$ diverges

<u>Theorem (6.2.2) (Limit Comparison Test)</u>

Let $f(x)$ and $g(x)$ be positive functions. If $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$, then

$$\int_a^{\infty} f(x)\ dx \text{ and } \int_a^{\infty} g(x)\ dx \text{ both converge or diverge.}$$

Example (4)

Show that $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^4}\ dx$ converges

Solution

Compare $\dfrac{x^2}{1+x^4}$ with $\dfrac{1}{1+x^2}$.

Science $\displaystyle\lim_{x \to \infty} \frac{\frac{x^2}{1+x^4}}{\frac{1}{1+x^2}} = \lim_{x \to \infty} \frac{x^2(1+x^2)}{(1+x^4)} = 1$, therefore

$\displaystyle\int_0^{\infty} \frac{x^2}{1+x^4}\ dx$ converges as $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\ dx$ is convergent.

Example (5)

Show that $\displaystyle\int_1^{\infty} \frac{1+2x^5}{1+x^6}\ dx$ diverges

Solution

Compare $\frac{1+2x^5}{1+x^6}$ with $\frac{1}{x}$.

Science $\lim_{x\to\infty}\frac{(1+2x^5)/(1+x^6)}{1/x}=\lim_{x\to\infty}\frac{x(1+2x^5)}{1+x^6}=2$, therefore

$\int_1^\infty \frac{1+2x^5}{1+x^6}\,dx$ diverges as $\int_1^\infty \frac{1}{x}\,dx$.

6.2.2 Improper Integrals of The Second Type

The improper integrals of the second type appears in either one of the following forms:

1. $\int_a^b f(x)\,dx$, where $f(x)$ is continuous on $(a, b]$ and is infinite at a.

The treatment of this integral is carried through the following limiting process:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f(x)\,dx$$

2. $\int_a^b f(x)\,dx$, where $f(x)$ is continuous on $[a, b)$ and is infinite at b.

The treatment of this integral is carried through the following limiting process:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_a^{b-\varepsilon} f(x)\,dx$$

3. $\int_a^b f(x)\,dx$, where $f(x)$ is continuous on $[a, b]$ except at $c\in(a, b)$ and it is infinite at c.

The treatment of this integral is carried through the following limiting process:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\varepsilon_1\to 0^+}\int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2\to 0^+}\int_{c+\varepsilon_2}^b f(x)\,dx$$

If anyone of the limits does not exists, the integral diverges.

However. If $\lim_{\varepsilon \to 0^+} \left(\int_a^{c-\varepsilon} f(x)\, dx + \int_{c+\varepsilon}^{b} f(x)\, dx \right)$ exists, we say that the *principal value* of the integral converges.

This means that approaching the point x = c from the right and the limit by different manners may lead to different results.

Example (6)

Evaluate $\int_1^5 \frac{1}{\sqrt{x-1}}\, dx$

Solution

$$\int_1^5 \frac{1}{\sqrt{x-1}}\, dx = \lim_{\varepsilon \to 0^+} \int_{1+\varepsilon}^{5} \frac{1}{\sqrt{x-1}}\, dx = \lim_{\varepsilon \to 0^+} \left(2\sqrt{x-1}\right)_{1+\varepsilon}^{5} = \lim_{\varepsilon \to 0^+} 2\left(\sqrt{4} - \sqrt{\varepsilon}\right) = 4$$

Therefore, the integral converges and is equal to 4.

Example (7)

Discuss whether the improper integral $\int_0^1 \frac{1}{1-x}\, dx$ converges or diverges

Solution

$$\int_0^1 \frac{1}{1-x}\, dx = \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{1}{1-x}\, dx = \lim_{\varepsilon \to 0^+} \left(-\ln(1-x)\right)_0^{1-\varepsilon} = \lim_{\varepsilon \to 0^+} \left(\ln 1 - \ln \varepsilon\right) = \infty$$

.

Since the limit does not exist, then the improper integral diverges.

Example (8)

Discuss whether the improper integral $\int_{-1}^{1} \frac{1}{x}\, dx$ converges or diverges

Solution

$$\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_0^1 \frac{dx}{x} = \lim_{\varepsilon_1 \to 0^+} \int_{-1}^{-\varepsilon_1} \frac{dx}{x} + \lim_{\varepsilon_2 \to 0^+} \int_{\varepsilon_2}^{1} \frac{dx}{x}$$

$$= \lim_{\varepsilon_1 \to 0^+} \left(\ln|x|\right)_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0^+} \left(\ln|x|\right)_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_1 \to 0^+} \left(\ln \varepsilon_1\right) + \lim_{\varepsilon_2 \to 0^+} \left(-\ln \varepsilon_2\right) = -\infty + \infty$$

Since both limits do not exist, therefore the Improper integral diverges.

However:

$$\lim_{\varepsilon\to 0^+} \int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} = \lim_{\varepsilon\to 0^+} \left(\ln|x|\right)_{-1}^{-\varepsilon} + \left(\ln|x|\right)_{\varepsilon}^{1} = 0$$

Therefore, the principal value of the improper integral converges.

Remark (1)

It must be noted that the domination comparison test can be applied also for the second type as follows.

Let $f(x)$ and $g(x)$ be continuous and let $0 \le f(x) \le g(x)$ for every $x \in [a, b]$. If $f(x)$ and $g(x)$ are infinite at $x = a$, then

1. If $\int_a^b g(x)\, dx$ converges, $\int_a^b f(x)\, dx$ also converges.

2. If $\int_a^b f(x)\, dx$ diverges, $\int_a^b g(x)\, dx$ also diverges.

Example (9)

Determine whether the improper integral $\int_0^3 \frac{\cosh}{(x-3)^2}\, dx$ converges or diverges

Solution

Since $\frac{\cosh x}{(x-3)^2} \ge \frac{1}{(x-3)^2}$ $\forall\ x\in[0, 3]$ and,

$$\int_0^3 \frac{1}{(x-3)^2}\, dx = \lim_{\varepsilon\to 0^+} \int_{-1}^{3-\varepsilon} \frac{1}{(x-3)^2}\, dx = \lim_{\varepsilon\to 0^+} \left(\frac{-1}{x-3}\right)_0^{3-\varepsilon} = \lim_{\varepsilon\to 0^+} \left(\frac{-1}{-\varepsilon} - \frac{1}{3}\right) = \infty$$

.

Hence the integral $\int_0^3 \frac{1}{(x-3)^2}\, dx$ diverges and so the integral $\int_0^3 \frac{\cosh x}{(x-3)^2}\, dx$ diverges.

Remark (2)

It is possible to face improper integrals which combine the two types. In this case the analysis which was performed separately for each type will be combined together as will be seen from the following examples:

Example (10)

Determine whether the improper integral $\int_0^\infty \frac{dx}{\sqrt{x}}$ converges or diverges

Solution

$$\int_0^\infty \frac{dx}{\sqrt{x}} = \lim_{\substack{\varepsilon\to 0^+ \\ r\to\infty}} \int_\varepsilon^r \frac{dx}{\sqrt{x}} = \lim_{\substack{\varepsilon\to 0^+ \\ r\to\infty}} \left(2\sqrt{x}\right)_\varepsilon^r = \lim_{\substack{\varepsilon\to 0^+ \\ r\to\infty}} \left(2\sqrt{r} - 2\sqrt{\in}\right) = \infty .$$

Therefore, the improper integral diverges.

Example (11)

If $p > 1$, show that the improper integral $\int_1^\infty \frac{dx}{x(\ln x)^p}$ converges or diverges

Solution

$$\int_1^\infty \frac{dx}{x(\ln x)^p} = \lim_{\substack{\varepsilon\to 0^+ \\ r\to\infty}} \int_{1+\varepsilon}^r \frac{dx}{x(\ln x)^p} = \lim_{\substack{\varepsilon\to 0^+ \\ r\to\infty}} \left(\frac{(\ln x)^{-p+1}}{-p+1}\right)_{1+\varepsilon}^r$$

$$= \lim_{\substack{\varepsilon\to 0^+ \\ r\to\infty}} \frac{1}{1-p}\left[\left(\ln r\right)^{1-p} - \left(\ln(1+\varepsilon)\right)^{1-p}\right]$$

Since $p > 1$, both terms tend to zero and hence the improper integral converges.

Exercises (6.2)

Determine whether each of the following improper integral converge or diverge. For those which converge, find their values whenever possible.

(1) $\int_{3}^{\infty} \frac{dx}{x \ln x}$ (2) $\int_{2}^{\infty} \frac{2dx}{x^2 - 1}$ (3) $\int_{0}^{\infty} \frac{x\,dx}{1 + x^2}$

(4) $\int_{-8}^{1} \frac{dx}{\sqrt[3]{x}}$ (5) $\int_{0}^{\infty} \frac{dx}{4 + x^2}$ (6) $\int_{-\infty}^{-2} \frac{2\ dx}{x^2 - 1}$

(7) $\int_{1}^{\infty} \frac{dx}{x^{5/3}}$ (8) $\int_{-\infty}^{-2} \frac{2\ dx}{x^2 + 4}$ (9) $\int_{-\infty}^{1} e^x\ dx$

(10) $\int_{1}^{\infty} \frac{2\ dx}{x^2 - x}$ (11) $\int_{-\infty}^{\infty} \frac{x\,dx}{1 + x^4}$ (12) $\int_{-\infty}^{\infty} \frac{2x\,dx}{(1 + x^2)^2}$

(13) $\int_{1}^{\infty} \frac{x\,dx}{(1 + x^2)^2}$ (14) $\int_{0}^{\infty} \frac{dx}{\sqrt{x}\,(1 + x)}$ (15) $\int_{2}^{\infty} \frac{x\,dx}{\sqrt{x^2 - 4}}$

(16) $\int_{-\infty}^{0} x\ e^x\ dx$ (17) $\int_{0}^{\infty} e^{-x} \cos x\ dx$ (18) $\int_{1}^{\infty} \frac{3}{e^x + 1}\ dx$

(19) $\int_{0}^{\infty} \frac{dx}{\sqrt{x^2 - 4}}$ (20) $\int_{0}^{\infty} \frac{dx}{\sqrt{x^6 - 4}}$ (21) $\int_{1}^{\infty} \frac{dx}{\sqrt{e^x - 1}}$

(22) $\int_{-8}^{1} \frac{dx}{\sqrt[3]{x}}$ (23) $\int_{0}^{\infty} \sin^2 x \cos x\ dx$ (24) $\int_{0}^{1} x^n\ dx$

(25) $\int_{0}^{3} \frac{dx}{\sqrt{9 - x^2}}$ (26) $\int_{0}^{2} \frac{dx}{\sqrt{4 - x^2}}$ (27) $\int_{0}^{1} x \ln x\ dx$

(28) $\int_{0}^{1} \frac{e^{-x}\ dx}{x^{2/3}}$ (29) $\int_{0}^{\pi/2} \tan x\ dx$ (30) $\int_{0}^{\pi} \frac{\sin x\ \ dx}{\sqrt{x}}$

(31) $\int_{-2}^{-1} \frac{dx}{x\sqrt{x^2 - 1}}$ (32) $\int_{-1}^{2} \frac{1}{x^2} \cos \frac{1}{x}\ dx$ (33) $\int_{0}^{1} \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

(34) $\int_{0}^{\pi/4} \frac{\sec x\ \ dx}{x^3}$ (35) $\int_{0}^{1} x^n\ \ln x\ dx$ (36) $\int_{-1}^{0} \frac{dx}{\sqrt[3]{x + 1}}$